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ON STABILITY OF PERTURBED SEMIGROUPS IN PARTIALLY ORDERED BANACH SPACES

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Abstract. We prove necessary and sufficient stability conditions for perturbed semigroups of linear operators in Banach spaces with cones and consider examples using these conditions. In particular, we consider an example where the boundary-value problem is perturbed by a linear operator with a delayed independent variable and establish stability conditions for such a perturbed semigroup.

CONTENTS

1. Introduction

Consider the following perturbed Cauchy problem in a Banach space:

$$\begin{cases} y' = -(\Gamma + M)y, \\ y(0) = y_0. \end{cases}$$

Here Γ is a closed operator (no boundedness is assumed), while M is a bounded operator treated as a perturbation. In the present paper, we show that the existence of a cone invariant with respect to Γ and M allows us to prove a link between the stability of the semigroup generated by this problem (i.e., the dissipativity of the operator $-\Gamma + M$) and the property of the operator $\Gamma^{-1}M$ to be contractive. More exactly, the following assertions are valid.

Theorem 3.1. *Let F be a real Banach space, $K \subset F$ be the reproducing cone, and the norm of the space F be monotone. Let the following linear operators be given*:

- *an operator $\Gamma : D(\Gamma) \subset F \to F$ is such that $-\Gamma$ is the generator of a strongly continuous semigroup such that $e^{-\Gamma t} \geq 0$ in the sense of K provided that $t \geq 0$;*
- *an operator $M : F \to F$ is bounded and $M \geq 0$ in the sense of K.*

Let the composition $\Gamma^{-1}M$ be completely continuous.

Then

$$\rho(e^{(-\Gamma+M)t}) < 1 \Longrightarrow \rho(\Gamma^{-1}M) < 1$$

for any positive t.

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Theorem 5.1. *Let F be a real Banach space and $K \subset F$ be the reproducing cone. Let the following linear operators be given:*

- *an operator $\Gamma : D(\Gamma) \subset F \to F$ is such that Γ^{-1} is completely continuous, $-\Gamma_C$ is the generator of an analytic and uniformly exponentially stable semigroup, and $e^{-\Gamma t}$ are positive in the sense of K provided that $t \geq 0$;*
- *an operator $M : F \to F$ is bounded and $M \geq 0$ in the sense of K.*

Then

$$\rho(\Gamma^{-1}M) < 1 \Longrightarrow \rho(e^{(-\Gamma+M)t}) < 1$$

for any positive t.

Secs. 3–5 contain proofs of these theorems.

Also, two examples using Theorem 5.1 are provided. The first example is described in Sec. 6; it refers to operators acting in the sequence space l_1 (the corresponding theory of infinite matrices can be found in [1, 14, 15]). We show that, using Theorem 5.1, one can obtain a stability condition for perturbed semigroups generated by such operators such that the obtained condition differs from the standard localization of eigenvalues by Geršgorin circles (see [15]).

Another example described in Sec. 7 is the Cauchy problem with the generator $(-\Gamma)(f) = f'' - af$ with the Neumann boundary-value conditions. All conditions of Theorem 5.1 are explicitly verified by computing of the Green function and verifying the positivity and analyticity of the obtained semigroup. Operators with delays are treated as perturbations; this is possible due to the conditions imposed on the operator M in Theorem 5.1 (the invariance of the cone and the estimate for the spectral radius or norm).

In Sec. 2, the used data about cones, semigroups, and the complexification of spaces are presented.

2. Preliminary Data

2.1. Cones in Banach spaces. In this section, we provide the main definitions related to cones in Banach space, used in the present paper (see, e.g., [8, 11]).

Definition 2.1. A closed subset K of a real Banach space X is called a *cone* if

(1) $\alpha \geq 0$, $x \in K \implies \alpha x \in K$;
(2) $x, y \in K \implies x + y \in K$;
(3) $K \cap (-K) = 0$.

Any cone K defines a *semiorder relation* in the space X. If $x, y \in X$ and $x - y \in K$, then it is denoted as follows: $x \geq y$.

Definition 2.2. A cone K is called *reproducing* if

$$\forall (x \in X) \exists (u, v \in K)[x = u - v].$$

Definition 2.3. A norm in a space X with cone K is called *monotone* if

$$0 \leq x \leq y \implies \|x\| \leq \|y\|.$$

Definition 2.4. An operator $A : X \to X$ is called *positive* (this is denoted as $A \geq 0$) if $A(K) \subset K$.

2.2. Semigroup theory. This section containing the main well-known notions and facts from the semigroup theory based on [3]. Partially, these data are contained in [2, 4, 9, 10, 13].

Definition 2.5. A family $\{T(t)\}_{t \geq 0}$ of bounded linear operators on a Banach space X is called a (one-parametric) *semigroup* on X if it satisfies the relation

$$\begin{cases} T(t+s) = T(t)T(s) \quad \forall (t, s \geq 0), \\ T(0) = I. \end{cases}$$

Definition 2.6. We say that a semigroup $\{T(t)\}_{t\geq 0}$ is *strongly continuous* or is a C_0*-semigroup* if the functions

$$\xi_x : \mathbb{R}_+ \to X$$

and

$$\xi_x : t \mapsto \xi_x(t) := T(t)x$$

are continuous for any x from X.

Definition 2.7. The *generator* of a strongly continuous semigroup $T(t)$ is the operator defined on the set

$$D(A) = \{x | \xi_x \text{ is differentiable}\}$$

and acting as follows:

$$A : x \mapsto \lim_{t\downarrow 0} \frac{T(t)x - x}{t}.$$

The semigroup $T(t)$ is frequently denoted by e^{At}.

Property 2.1. *For any strongly continuous semigroup, its generator is linear, closed, and densely defined.*

Property 2.2. *If* $\{T(t)\}_{t\geq 0}$ *is a strongly continuous semigroup, then there exist a real* ω *and* M *from* $(1, +\infty)$ *such that*

$$\|T(t)\| \leq Me^{\omega t}$$

for any nonnegative t.

Definition 2.8. Let $\mathcal{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup. Then the number

$$\omega_0(\mathcal{T}) = \inf \left\{\omega \in \mathbb{R} : \exists (M_\omega) \forall (t \geq 0) \left[\|T(t)\| \leq M_\omega e^{\omega t}\right]\right\}$$

is called its *growth order*.

Property 2.3. *If* $\mathcal{T} = \{T(t)\}_{t\geq 0}$ *is a strongly continuous semigroup, then*

$$\rho(T(t)) = e^{\omega_0(\mathcal{T})t}$$

for any nonnegative t*, where* ρ *is the spectral radius.*

Definition 2.9. We say that a semigroup $\{T(t)\}_{t\geq 0}$ is *uniformly exponentially stable* if

$$\exists (\varepsilon > 0) \left[\lim_{t\to\infty} e^{\varepsilon t}\|T(t)\| = 0\right].$$

Property 2.4. *A strongly continuous semigroup* $\{T(t)\}_{t\geq 0}$ *is uniformly exponentially stable if and only if its growth order* $\omega_0(\mathcal{T})$ *is negative.*

Definition 2.10. Let X be a Banach space and $A : D(A) \subset A \to X$ be a closed operator. Then the number

$$s(A) := \sup\{\operatorname{Re} \lambda \in \sigma(A)\}$$

is called the *spectral boundary.*

Definition 2.11. If $\mathcal{T} = \{T(t)\}_{t\geq 0}$ is a semigroup generated by a generator A, then the number

$$\omega_{ess}(\mathcal{T}) = \omega_{ess}(A) := \inf_{t>0} \frac{1}{t} \ln \|T(t)\|_{ess},$$

where

$$\|D\|_{ess} = \inf\{\|D - K\| : K \text{ is a compact operator}\}$$

for any operator D, is called the *essential growth order* of that semigroup.

Property 2.5. *Let* A *be the generator of a strongly continuous semigroup* $\mathcal{T} = \{T(t)\}_{t\geq 0}$*. Then*

$$\omega_0(\mathcal{T}) = \max\{\omega_{ess}(\mathcal{T}), s(A)\}.$$

Property 2.6. *Let A be the generator of a strongly continuous semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$. Then*

$$-\infty \leq s(A) \leq \omega_0(\mathcal{T}) = \inf_{t>0} \frac{1}{t} \ln \|T(t)\| = \lim_{t\to\infty} \frac{1}{t} \ln \|T(t)\| = \frac{1}{t_0} \ln \rho(T(t_0)) < +\infty$$

for any positive t_0.

Property 2.7. *Let A be the generator of a strongly continuous semigroup $\mathcal{T} = \{T(t)\}_{t\geq 0}$ and K be a compact operator. Then*

$$\omega_{ess}(A) = \omega_{ess}(A + K).$$

Introduce and describe analytic semigroups.

Definition 2.12. Introduce the notation

$$\Sigma_\delta = \{z \in \mathbb{C} : |\arg(z)| < \delta\} \setminus \{0\}.$$

A family of bounded linear operators $\{T(z)\}_{z\in\Sigma_\delta\cup\{0\}}$ acting in a Banach space X is called an *analytic semigroup* (of angle δ from $(0, \pi/2]$) if

(1) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for all z_1 and z_2 from Σ_δ;
(2) the map $z \mapsto T(z)$ is analytic in Σ_δ;
(3) $\lim\limits_{\Sigma'_\delta\to 0} T(z)x = x$ for any x from X and any δ' from $(0, \delta)$.

Lemma 2.1. *Let $A : D(A) \subset F \to F$ be a linear operator in a complex Banach space F. If $\vartheta \in \mathbb{R}$, then*

$$e^{-i\vartheta}\sigma(A) = \sigma(e^{i\vartheta}A).$$

Lemma 2.2. *Let A be the generator of an analytic semigroup of angle α and ϑ belong to $(-\alpha, \alpha)$. Then $e^{A(e^{i\vartheta}t)}$ is a strongly continuous semigroup and its generator A_ϑ coincides with $e^{i\vartheta}A$.*

2.3. Complexification of Spaces and Operators. For spaces and operators, we use the notion of complexification described in [5, Sec. II.XIII.2].

For any arbitrary linear operator $A : F \to F$, where F is a real Banach space, one can construct an operator $A_C : F_C \to F_C$, where $F_C = F \times F$ is a complex Banach space with the norm

$$\|(x, y)\| = \max_{\theta\in[0,2\pi]} \|x \cos\theta + y \sin\theta\|$$

and $A_C : (x, y) \mapsto (Ax, Ay)$. The following properties are easily verified.

Property 2.8. $A_C B_C = (AB)_C$ *and* $(A^{-1})_C = (A_C)^{-1}$.

Property 2.9. *For all real α and β, the relation $(\alpha A + \beta B)_C = \alpha A_C + \beta B_C$ holds.*

Property 2.10. *The operator A_C is compact if and only if the operator A is compact.*

Property 2.11. $\|A_C\| = \|A\|$ *due to the Gelfand relation $\rho(A_C) = \rho(A)$.*

Property 2.12. *The semigroup e^{At} is strongly continuous if and only if the semigroup $(e^{At})_C$ is strongly continuous; if this is valid, then the reproducing operator $(e^{At})_C$ coincides with A_C, i.e.,*

$$e^{A_C t} = (e^{At})_C.$$

Also,

$$\omega_0(e^{A_C t}) = \omega_0(e^{At})$$

due to Property 2.11.

The spectrum, spectral radius, and eigenvalues of an operator A defined over a real Banach space are the spectrum, spectral radius, and eigenvalues (respectively) of the operator A_C.

3. Stability: Necessary Condition

Theorem 3.1. *Let F be a real Banach space, $K \subset F$ be a reproducing cone, and the norm of the space F be monotone. Let the following linear operators be given:*

- *an operator $\Gamma : D(\Gamma) \subset F \to F$ is such that $-\Gamma$ is the generator of a strongly continuous and uniformly exponentially stable semigroup and $e^{-\Gamma t} \geq 0$ in the sense of K for all nonnegative t;*
- *an operator $M : F \to F$ is bounded and $M \geq 0$ in the sense of K.*

Let the composition $\Gamma^{-1}M$ be completely continuous. Then

$$\rho(e^{(-\Gamma+M)t}) < 1 \Longrightarrow \rho(\Gamma^{-1}M) < 1$$

for any positive t.

Proof. By virtue of the compactness, the spectrum of $\Gamma^{-1}M$ consists of isolated eigenvalues of finite multiplicity (see [6, III.6.7]). If $\Gamma^{-1}M$ has no nonzero eigenvalues, then $\rho(\Gamma^{-1}M) < 1$, which completes the proof. Consider the case where $\Gamma^{-1}M$ has nonzero eigenvalues. Assume the converse, i.e., $\rho(\Gamma^{-1}M) \geq 1$, and note that

$$\Gamma^{-1} = R(0, \Gamma) = -R(0, -\Gamma) = \int\limits_0^{+\infty} e^{-\Gamma t} dt \geq 0.$$

Then the operator $\Gamma^{-1}M$ is positive (as a composition of positive operators) and compact (by hypothesis). Since K is a reproducing cone, it follows from [11, Th. 6.1, Sec. 6] that there exists an eigenvector x_0 from K corresponding to the eigenvalue $\lambda = \rho(\Gamma^{-1}M)$, i.e.,

$$\Gamma^{-1}Mx_0 = \lambda x_0,$$

which means that

$$x_0 = \frac{1}{\lambda}\Gamma^{-1}Mx_0$$

and

$$\Gamma x_0 = \Gamma\frac{1}{\lambda}\Gamma^{-1}Mx_0 = \frac{1}{\lambda}\Gamma\Gamma^{-1}Mx_0 = \frac{1}{\lambda}Mx_0 \in K.$$

Then

$$(-\Gamma + M)x_0 = Mx_0 - \Gamma x_0 = \Gamma\Gamma^{-1}Mx_0 - \Gamma x_0 = \Gamma\lambda x_0 - \Gamma x_0 = \lambda\Gamma x_0 - \Gamma x_0 = (\lambda - 1)\Gamma x_0 \in K.$$

Consider the Cauchy problem

$$\begin{cases} y' = (-\Gamma + M)y \\ y(0) = x_0 \end{cases} \tag{3.1}$$

and its solution $y(t) = e^{(-\Gamma+M)t}x_0$. For all positive t and $\triangle t$, we have the relation

$$\begin{aligned} y(t + \triangle t) - y(t) &= e^{(-\Gamma+M)t}e^{(-\Gamma+M)\triangle t}x_0 - e^{(-\Gamma+M)t}x_0 = e^{(-\Gamma+M)t}(e^{(-\Gamma+M)\triangle t}x_0 - x_0) \\ &= e^{(-\Gamma+M)t}\int\limits_0^{\triangle t} e^{(-\Gamma+M)s}(-\Gamma + M)x_0\, ds = e^{(-\Gamma+M)t}\int\limits_0^{\triangle t} e^{(-\Gamma+M)s}(\lambda - 1)\Gamma x_0\, ds. \end{aligned} \tag{3.2}$$

Note that the operator $e^{(-\Gamma+M)t}$ is positive for any nonnegative t because the iterative scheme $v_0 = I$, $v_{i+1} = Qv_i$ with the positive operator

$$(Qv)(t) = e^{-\Gamma t} + \int\limits_0^t e^{(-\Gamma)(t-s)}Mv(s)ds$$

converges to it. Due to the converse assumption, we have $\lambda = \rho(\Gamma^{-1}M) \geq 1$. Therefore, (3.2) implies that

$$y(t + \triangle t) - y(t) \geq 0,$$

i.e.,

$$y(t + \triangle t) \geq y(t)$$

in the sense of K. By virtue of the positivity of $y(t + \triangle t)$ and $y(t)$ and the monotonicity of the norm, this implies the inequality

$$\|y(t + \triangle t)\| \geq \|y(t)\|$$

contradicting the semigroup stability implied by the inequality $\rho(e^{(-\Gamma+M)t}) < 1$. Therefore, our assumption is wrong and $\rho(\Gamma^{-1}M) < 1$. □

4. Positiving Multivalued Maps

Let F be a real Banach space and $K \subset F$ be a reproducing cone. Then the following multivalued map can be defined on F:

$$P : F \longrightarrow 2^K,$$
$$P : x \longmapsto \{u + v | u, v \in K; u - v = x\},$$

where 2^K is the set of all subsets of the cone K. Since the cone is reproducing, it follows that $P(x)$ contains at least one element for any x from F.

Deduce several properties of the map P.

Property 4.1. $\forall(x \in F)[P(x) \subset K]$.

This is obvious because any y from $P(x)$ is a sum of two elements of K.

Property 4.2. $\forall(x \in F)[P(x) = P(-x)]$.

Let $y \in P(x)$. Then $\exists(u, v \in K)[(u + v = y) \wedge (u - v = x)]$. This implies that $v - u = -x \Rightarrow y = u + v \in P(-x)$, i.e., $P(x) \subset P(-x)$. Taking the vector $-x$ for x, we see that the inverse embedding holds.

Property 4.3. $\forall(x \in F, y \in P(x))[(y + x \in K) \wedge (y - x \in K)]$.

$y \in P(x) \overset{\text{def}}{\Leftrightarrow} \exists(u, v \in K)[(u + v = y) \wedge (u - v = x)]$. Then $y + x = u + v + u - v = 2u \in K$ and $y - x = u + v - (u - v) = u + v - u + v = 2v \in K$.

Property 4.4. $y \geq x, y \geq -x \Longrightarrow y \in P(x)$.

The claimed decomposition is as follows: $u = \dfrac{1}{2}(y + x) \in K$, $v = \dfrac{1}{2}(y - x) \in K$, $u + v = y$, and $u - v = x$.

Property 4.5. $P(0) = K$.

Due to Property 4.1, $P(0)$ is a subset of K. To prove the inverse embedding, assume that $y \in K$. Then $u = v = \dfrac{y}{2}$ is the claimed decomposition of the vector 0.

Property 4.6. $\forall(x \in F)[b \in P(x), a \geq b \Longrightarrow a \in P(x)]$

$b \in P(x) \overset{\text{def}}{\Leftrightarrow} \exists(u, v \in K)[(u + v = b) \wedge (u - v = x)]$,
$a \geq b \Longrightarrow a - b \in K$.

Let $u_1 = u + \dfrac{a - b}{2} \in K$ and $v_1 = v + \dfrac{a - b}{2} \in K$. Then $u_1 + v_1 = u + \dfrac{a - b}{2} + v + \dfrac{a - b}{2} = b + a - b = a$ and $u_1 - v_1 = u + \dfrac{a - b}{2} - \left(v + \dfrac{a - b}{2}\right) = u - v = x$. Therefore, $a \in P(x)$.

Property 4.7. $\forall(x \in F)[P(x)$ *is a closed set*].

Let $\{y_i\}$ be an arbitrary sequence such that $\{y_i\} \subset P(x)$ and $y_i \to y$. Then $\frac{1}{2}(y_i + x) \to \frac{1}{2}(y + x)$ and $\frac{1}{2}(y_i - x) \to \frac{1}{2}(y - x)$. By virtue of the closedness of the cone and Property 4.3, we have the relations $u = \frac{1}{2}(y + x) \in K$, $v = \frac{1}{2}(y - x) \in K$, $u + v = y$, and $u - v = x$. Hence, $y \in P(x)$ and $P(x)$ is closed.

Property 4.8. $\forall(x \in F, \alpha \in \mathbb{R})[|\alpha|P(x) \subset P(\alpha x)]$

For $\alpha = 0$, we have $|\alpha|P(x) = 0 \subset P(\alpha x) = P(0) = K$.

For $\alpha > 0$, we have $|\alpha|P(x) = \alpha P(x)$. Let $y \in P(x)$, i.e., $\exists(u, v \in K)[(u + v = y) \wedge (u - v = x)]$. Then $\alpha y = \alpha(u + v) = \alpha u + \alpha v$, $\alpha u \in K$, and $\alpha v \in K$. Since $\alpha u - \alpha v = \alpha(u - v) = \alpha x$, it follows that $\alpha y \in P(\alpha x)$.

For $\alpha < 0$, we have $|\alpha|P(x) = -\alpha P(x)$. Let $y \in P(x)$, i.e., $\exists(u, v \in K)[(u+v = y)\wedge(u-v = x)]$. Then $-\alpha y = -\alpha(u+v) = (-\alpha u)+(-\alpha v)$, $-\alpha u \in K$, and $-\alpha v \in K$. Since $(-\alpha v)-(-\alpha u) = \alpha(u-v) = \alpha x$, it follows that $-\alpha y \in P(\alpha x)$.

Property 4.9. $\forall(x_1, x_2 \in F)[P(x_1) + P(x_2) \subset P(x_1 + x_2)]$.

Let $y_1 \in P(x_1)$ and $y_2 \in P(x_2)$, i.e., $\exists(u_1, v_1 \in K)[(u_1 + v_1 = y_1) \wedge (u_1 - v_1 = x_1)]$ and $\exists(u_2, v_2 \in K)[(u_2 + v_2 = y_2) \wedge (u_2 - v_2 = x - 2)]$. Then $y_1 + y_2 = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2)$ and $(u_1 + u_2) - (v_1 + v_2) = u_1 - v_1 + u_2 - v_2 = x_1 + x_2$. Therefore, $y_1 + y_2 \in P(x_1 + x_2)$, which means that $P(x_1) + P(x_2) \subset P(x_1 + x_2)$.

Property 4.10. *Let* $S : F \to F$ *be a linear positive operator. Then* $\forall(x \in F)[SPx \subset PSx]$, *where*

$$P(x) \stackrel{\text{def}}{=} \{u + v | u, v \in K; u - v = x\}$$

and

$$\begin{aligned} SP(x) &= S\{u + v | u, v \in K; u - v = x\} \\ &= \{Su + Sv | u, v \in K; u - v = x\} \subset P(Su - Sv) = PS(u - v) = PS(x). \end{aligned}$$

5. Stability Preservation Theorem

Lemma 5.1. *Let* F *be a complex Banach space and the following linear operators be given*:

- *an operator* $\Gamma : D(\Gamma) \subset F \to F$ *is such that* Γ^{-1} *is a compact operator,*
- $M : F \to F$ *is a bounded operator.*

Also, let $\rho(\Gamma^{-1}M) < 1$. *Then the resolvent of the operator* $-\alpha\Gamma + M$ *is compact provided that* $\alpha' \leq \alpha < +\infty$ *and* $0 < \alpha' < 1$.

Proof. Take ε satisfying the inequality $0 < \varepsilon < 1 - \rho(\Gamma^{-1}M)$. In F, one can introduce a norm $\|\cdot\|_\varepsilon$ such that it is equivalent to the current norm and the corresponding operator norm is as follows:

$$\|\Gamma^{-1}M\|_\varepsilon \leq \rho(\Gamma^{-1}M) + \varepsilon < \rho(\Gamma^{-1}M) + 1 - \rho(\Gamma^{-1}M) = 1$$

(see [8, Ch. 2, Sec. 5.2]).

Now, select α' such that $\|\Gamma^{-1}M\|_\varepsilon < \alpha' < 1$. One can see that the estimate $\left\|\frac{1}{\alpha}\Gamma^{-1}M\right\|_\varepsilon < 1$ holds for any α satisfying the inequality $\alpha' \leq \alpha < +\infty$. Then fix λ such that

$$0 < \lambda < \frac{1 - \|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon}{\|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon}.$$

Then

$$\|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon + \|\frac{\lambda}{\alpha}\Gamma^{-1}\|_\varepsilon < 1,$$

which means that

$$\rho\left(\frac{1}{\alpha}\Gamma^{-1}M+\frac{\lambda}{\alpha}\Gamma^{-1}\right)\le\|\frac{1}{\alpha}\Gamma^{-1}M+\frac{\lambda}{\alpha}\Gamma^{-1}\|_\varepsilon\le\|\frac{1}{\alpha}\Gamma^{-1}M\|_\varepsilon+\|\frac{\lambda}{\alpha}\Gamma^{-1}\|_\varepsilon<1.$$

Therefore, the resolvent can be expanded into the Neumann series:

$$\begin{aligned} R_\lambda(-\alpha\Gamma+M)&=(-\alpha\Gamma+M-\lambda I)^{-1}=\left(-\alpha\Gamma\left(I-\frac{1}{\alpha}\Gamma^{-1}M-\frac{\lambda}{\alpha}\Gamma^{-1}\right)\right)^{-1}\\ &=\left(I-\frac{1}{\alpha}\Gamma^{-1}M-\frac{\lambda}{\alpha}\Gamma^{-1}\right)^{-1}\left(-\frac{1}{\alpha}\Gamma^{-1}\right)=\sum_{k=0}^{\infty}\left(\frac{1}{\alpha}\Gamma^{-1}M+\frac{\lambda}{\alpha}\Gamma^{-1}\right)^k\left(-\frac{1}{\alpha}\Gamma^{-1}\right). \end{aligned}\tag{5.1}$$

From (5.1), it follows that $R_\lambda(-\alpha\Gamma+M)$ is compact, i.e., $-\alpha\Gamma+M$ is an operator with a compact resolvent (see [6, Ch. III, Sec. 6.8]) for any α such that $\alpha'\le\alpha<+\infty$, where $\alpha'<1$. □

Lemma 5.2. *Let F be a real Banach space and $K\subset F$ be a reproducing cone. Let the following linear operators be given*:

- *an operator $\Gamma: D(\Gamma)\subset F\to F$ is such that $-\Gamma$ is the generator of a strongly continuous uniformly exponentially stable semigroup and this semigroup is positive in the sense of K for any value of the parameter*;
- *an operator $M: F\to F$ is bounded and positive in the sense of K.*

Let $\rho(\Gamma^{-1}M)<1$. Then an operator of the kind $-\alpha\Gamma+M$ might have imaginary eigenvalues only if $\alpha<1$.

Proof. Assume the converse: the operator $\alpha_0\Gamma+M$ has an eigenvalue $i\omega$, $\omega\in\mathbb{R}$, an eigenvector x_0+iy_0 corresponds to it, and $\alpha_0\ge1$.

The function

$$y_1(t)=(x_0+iy_0)(\cos\omega t+i\sin\omega t)=(x_0\cos\omega t-y_0\sin\omega t)+i(x_0\sin\omega t+y_0\cos\omega t)$$

satisfies the equation

$$y'=(-\alpha_0\Gamma+M)y$$

and the initial-value condition $y(0)=x_0+iy_0$, while the function

$$y_2(t)=(x_0-iy_0)(\cos\omega t-i\sin\omega t)=(x_0\cos\omega t-y_0\sin\omega t)-i(x_0\sin\omega t+y_0\cos\omega t)$$

satisfies it and the initial-value condition $y(0)=x_0-iy_0$. Hence, their sum

$$y^*(t)=2x_0\cos\omega t-2y_0\sin\omega t$$

satisfies it as well (under the initial-value condition $y(0)=2x_0$).

The function y^* is a periodic solution (the period T is equal to $\frac{2\pi}{\omega}$). It is a nondegenerate solution because the eigenvector x_0+iy_0 is not equal to $0+i0$. This solution lies in the real component F_C. Therefore, to continue the proof of this lemma, it suffices to deal with operators acting in F (instead of operators acting in F_C).

Since y^* is a solution of the inhomogeneous system

$$\begin{cases} y'=-\alpha_0\Gamma y+My^*\\ y(0)=2x_0 \end{cases},$$

it follows that

$$y^*(t)=\int_{-\infty}^{t}e^{-\alpha_0\Gamma(t-s)}My^*(s)ds.$$

Consider the following family of linear operators:

$$Q_t: F\longrightarrow F,$$

$$Q_t : x \longmapsto \int\limits_{-\infty}^{t} e^{-\alpha_0 \Gamma(t-s)} M x ds.$$

Let us show that all the operators Q_t coincide with the operator $Q \stackrel{\text{def}}{=} \frac{1}{\alpha_0}\Gamma^{-1}M$. For any x from F, the following relation is valid:

$$Q_t x = \int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)} M x ds = -\int\limits_{+\infty}^{0} e^{-\alpha_0\Gamma q} M x dq = \int\limits_{0}^{+\infty} e^{-\alpha_0\Gamma q} M x dq = -R(0, -\alpha_0\Gamma) M x = \frac{1}{\alpha_0}\Gamma^{-1} M x,$$

i.e.,

$$Q_t = \frac{1}{\alpha_0}\Gamma^{-1} M = Q.$$

Note that $\alpha_0 \geq 1$ by assumption and $\rho(\Gamma^{-1}M) < 1$. Hence, one can introduce a norm $\|\cdot\|_L$ in F such that it is equivalent to the norm given in F and

$$\|\Gamma^{-1}M\|_L \stackrel{\text{def}}{=} \sup_{x \in F, x \neq 0} \frac{\|\Gamma^{-1}Mx\|_L}{\|x\|_L} < 1$$

(see [8, Ch. 2, Sec. 5.2]).

Then

$$\|Q_t x - Q_t y\|_L = \|Q_t(x-y)\|_L = \|\frac{1}{\alpha_0}\Gamma^{-1}M(x-y)\|_L \leq \frac{1}{\alpha_0}\|\Gamma^{-1}M\|_L \|x-y\|_L$$

for all x and y from F.

Hence, Q_t is a contraction map with a unique fixed point in F. Since Q_t is a linear operator, it follows that the fixed point is 0.

Since K is a reproducing cone, it follows that the map P can be defined on F. Consider the set

$$\widetilde{P} \stackrel{\text{def}}{=} \bigcap_{s \in \mathbb{R}} P(y^*(s)).$$

Each set $P(y^*(s))$ is closed. Therefore, the set $\widetilde{P}$ is closed as well (as the intersection of a finite number of closed sets).

To show that the set $\widetilde{P}$ is not empty, introduce the notation

$$a_1 = 2x_0 + 2y_0, \quad a_2 = 2x_0 - 2y_0.$$

Then

$$2x_0 = \frac{a_1 + a_2}{2}, \quad 2y_0 = \frac{a_1 - a_2}{2}.$$

Since K is a reproducing cone, it follows that there exist its elements u_1, v_1, u_2, and v_2 such that

$$a_1 = u_1 - v_1, \quad a_2 = u_2 - v_2.$$

Introduce the notation $\xi = u_1 + v_1 + u_2 + v_2$. It is obvious that $\xi \in K$ and

$$\begin{aligned}
\xi - y^*(t) &= u_1 + v_1 + u_2 + v_2 - 2x_0 \cos\omega t + 2y_0 \sin\omega t \\
&= u_1 + v_1 + u_2 + v_2 - \frac{a_1 + a_2}{2}\cos\omega t + \frac{a_1 - a_2}{2}\sin\omega t \\
&= u_1 + v_1 + u_2 + v_2 - \frac{u_1 - v_1 + u_2 - v_2}{2}\cos\omega t + \frac{u_1 - v_1 - u_2 + v_1}{2}\sin\omega t \\
&= u_1\left(1 - \frac{\cos\omega t}{2} + \frac{\sin\omega t}{2}\right) + v_1\left(1 + \frac{\cos\omega t}{2} - \frac{\sin\omega t}{2}\right) + u_2\left(1 - \frac{\cos\omega t}{2} - \frac{\sin\omega t}{2}\right) \\
&\quad + v_2\left(1 + \frac{\cos\omega t}{2} + \frac{\sin\omega t}{2}\right).
\end{aligned}$$

One can see that each vector is multiplied by a nonnegative factor. Therefore, $\xi - y^*(t) \in K$, i.e., $\xi \geq y^*(t)$ for any t. Since $-y^*(t) = y^*\Big(t + \dfrac{\pi}{\omega}\Big)$, it follows from Property 4.4 that $\xi \in P(y^*(t))$, i.e., $\xi \in \widetilde{P}$.

To continue the investigation of the operators Q_t, we note that, for any real t and any $\widetilde{y}$ from $\widetilde{P}$, we have the relation

$$Q_t\widetilde{y} = \int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}M\widetilde{y}ds = \int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}M\left(\frac{\widetilde{y}+y^*(s)}{2} + \frac{\widetilde{y}-y^*(s)}{2}\right)ds$$

$$= \int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}M\frac{\widetilde{y}+y^*(s)}{2}ds + \int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}M\frac{\widetilde{y}-y^*(s)}{2}ds.$$

Due to the nonnegativity of those integrals, the last sum belongs to

$$P\left(\int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}M\frac{\widetilde{y}+y^*(s)}{2}ds - \int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}M\frac{\widetilde{y}-y^*(s)}{2}ds\right)$$

$$= P\left(\int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}M\left(\frac{\widetilde{y}+y^*(s)}{2} - \frac{\widetilde{y}-y^*(s)}{2}\right)ds\right) = P\left(\int\limits_{-\infty}^{t} e^{-\alpha_0\Gamma(t-s)}My^*(s)ds\right) = P(y^*(t)).$$

Since $Q_t = Q$ for any t, it follows that $Q\widetilde{y} \in P(y^*(t))$ for any real t, i.e., $Q\widetilde{y} \in \bigcap\limits_{t\in\mathbb{R}} P(y^*(t)) = \widetilde{P}$. Hence, the map

$$\widetilde{Q} : \widetilde{P} \longrightarrow \widetilde{P}, \quad \widetilde{Q} : x \longmapsto \frac{1}{\alpha_0}\Gamma^{-1}Mx,$$

is well defined. Since the set $\widetilde{P}$ is closed, it follows that it can be treated as a metric space with the metric generated by the norm of the space F. The map $\widetilde{Q}$ is contractive. Therefore, it has a fixed point y_f from $\widetilde{P}$. Then y_f is a fixed point of the map Q, while the only fixed point of Q is 0. Therefore, $y_f = 0$ and $0 \in \widetilde{P}$.

Note that $\widetilde{P}$ is a subset of $P(y^*(0))$. Therefore, 0 belongs to $P(y^*(0))$. This means that $\exists(u_x, v_x \in K : u_x + v_x = 0)[u_x - v_x = y^*(0)]$, i.e.,

$$u_x - v_x = y^*(0) = 2x_0\cos(\omega 0) - 2y_0\sin(\omega 0) = 2x_0.$$

Since K is a cone, it follows that $(u_x, v_x \in K) \wedge (u_x + v_x = 0) \Longrightarrow (u_x = 0) \wedge (v_x = 0)$. Then $u_x - v_x = 0 - 0 = 0 = 2x_0 = x_0$.

In the same way, $\widetilde{P} \subset P\Big(y^*\Big(\dfrac{\pi}{2\omega}\Big)\Big) \Longrightarrow 0 \in P\Big(y^*\Big(\dfrac{\pi}{2\omega}\Big)\Big) \Longrightarrow \exists(u_y, v_y \in K : u_y + v_y = 0)[u_y - v_y = y^*\Big(\dfrac{\pi}{2\omega}\Big) = -2y_0] \Longrightarrow (u_y = 0) \wedge (v_y = 0) \Longrightarrow u_y - v_y = 0 = -2y_0 = y_0$.

Thus, both x_0 and y_0 are equal to zero. However, $x_0 + iy_0$ is an eigenvector and it cannot be equal to the zero vector.

We obtain a contradiction. Therefore, our assumption is wrong and $\alpha_0 < 1$. $\square$

Theorem 5.1. *Let F be a real Banach space and $K \subset F$ be a reproducing cone. Let the following linear operators be given*:

- *an operator $\Gamma : D(\Gamma) \subset F \to F$ is such that Γ^{-1} is completely continuous, $-\Gamma_C$ is the generator of an analytic and exponentially stable semigroup, and $e^{-\Gamma t}$ is positive in the sense of K for any nonnegative t*;
- *an operator $M : F \to F$ is bounded and $M \geq 0$ in the sense of K.*

Then

$$\rho(\Gamma^{-1}M) < 1 \Longrightarrow \rho(e^{(-\Gamma+M)t}) < 1$$

for any positive t.

Proof. Since $e^{-\Gamma t}$ is a strongly continuous semigroup, it follows that there exist c and ω_1 such that $\|e^{-\Gamma t}\| \le ce^{\omega_1 t}$ and $\omega_1 < 0$ (because the semigroup is stable).

Let α be nonnegative. Then $D(-\alpha\Gamma) = D(-\Gamma)$, $e^{(-\alpha\Gamma)t} = e^{-\Gamma(\alpha t)}$ is a strongly continuous semigroup, and $\|e^{(-\alpha\Gamma)t}\| \le ce^{\omega_1 \alpha t}$.

It follows from Lemma 5.1 that there exists α' such that $0 < \alpha' < 1$ and $-\alpha\Gamma_C + M_C$, i.e., $-\alpha\Gamma + M$ is an operator with a compact resolvent provided that $\alpha \ge \alpha'$.

Investigate the operator family

$$\{-\alpha\Gamma + M\}_{\alpha \ge \alpha'} \tag{5.2}$$

in detail.

Since M is a bounded operator, it follows from [3, Theorem III.1.3, p. 158] that $e^{(-\alpha\Gamma+M)t}$ is a strongly continuous semigroup and

$$\|e^{(-\alpha\Gamma+M)t}\| \le ce^{(\alpha\omega_1 + c\|M\|)t}.$$

Since $\omega_1 < 0$, it follows that if $\alpha > -\frac{c\|M\|}{\omega_1}$, then $\alpha\omega_1 + c\|M\| < 0$, i.e., the spectrum $\sigma(-\alpha\Gamma + M)$ is located in the left-hand half-plane.

Since $-\Gamma_C$ is the generator of an analytic semigroup, it follows that there exists ϑ from $(0, \pi/2)$ such that the restrictions of $e^{-\Gamma_C z}$ to $\{e^{i\vartheta}t \,|\, t \ge 0\}$ and $\{e^{-i\vartheta}t \,|\, t \ge 0\}$ are strongly continuous semigroups. Due to Lemma 2.2, their generators are $-e^{i\vartheta}\Gamma_C$ and $-e^{-i\vartheta}\Gamma_C$, respectively. Then there exist M_i, M_{-i}, ω_i, and ω_{-i} such that

$$\|e^{(-e^{i\vartheta}\Gamma_C)t}\| \le M_i e^{\omega_i t}, \quad \|e^{(-e^{-i\vartheta}\Gamma_C)t}\| \le M_{-i} e^{\omega_{-i} t}.$$

Therefore,

$$\|e^{(e^{i\vartheta}(-\alpha\Gamma_C + M))t}\| \le M_i e^{(\alpha\omega_i + M_i\|M\|)t}, \quad \|e^{(e^{-i\vartheta}(-\alpha\Gamma_C + M_C))t}\| \le M_{-i} e^{(\alpha\omega_{-i} + M_{-i}\|M\|)t}.$$

This means that

$$\operatorname{Re}\sigma(e^{i\vartheta}(-\alpha\Gamma_C + M_C)) \le \alpha\omega_i + M_i\|M\|, \quad \operatorname{Re}\sigma(e^{-i\vartheta}(-\alpha\Gamma_C + M_C)) \le \alpha\omega_{-i} + M_{-i}\|M\|.$$

Then, due to Lemma 2.1, we have the inequalities

$$\operatorname{Re} e^{-i\vartheta}\sigma(-\alpha\Gamma + M) \le \alpha\omega_i + M_i\|M\|, \quad \operatorname{Re} e^{i\vartheta}\sigma(-\alpha\Gamma + M) \le \alpha\omega_{-i} + M_{-i}\|M\|,$$

i.e., if $\lambda \in \sigma(-\alpha\Gamma + M)$, then

$$\operatorname{Re}\lambda\cos\vartheta + \operatorname{Im}\lambda\sin\vartheta \le \alpha\omega_i + M_i\|M\|, \quad \operatorname{Re}\lambda\cos\vartheta - \operatorname{Im}\lambda\sin\vartheta \le \alpha\omega_{-i} + M_{-i}\|M\|.$$

Finally,

$$\operatorname{Im}\lambda \le -\cot\vartheta \operatorname{Re}\lambda + \frac{\alpha\omega_i + M_i\|M\|}{\sin\vartheta} \tag{5.3}$$

and

$$\operatorname{Im}\lambda \ge \cot\vartheta \operatorname{Re}\lambda - \frac{\alpha\omega_{-i} + M_{-i}\|M\|}{\sin\vartheta}. \tag{5.4}$$

Thus, the spectrum of $-\alpha\Gamma + M$ is located inside a sector of the complex plane such that the direction of its extending part coincides with the negative direction of the real axis.

On the other hand, $-\alpha\Gamma + M$ is an operator with a compact resolvent provided that $\alpha \ge \alpha'$ (see above). Therefore, for any fixed α, its spectrum consists of isolated eigenvalues of finite multiplicity; if α varies, then the eigenvalues form continuous branches $\mu(\alpha)$ such that their intersections are possible and it is possible that $|\mu(\alpha)| \to \infty$ as α tends to a finite value (for details, see [6] and, in particular, Theorem IV.3.16 of this book). Family (5.2) is holomorphic of type (A) and its families of eigenvalues are analytic.

Let a function $\mu(\alpha)$ acting from (a, b) or $[a, b)$ to $\mathbb{C}$ be any branch of an eigenvalue such that $\alpha' \leq a < 1$ and $1 < b \leq +\infty$. Consider the following possible cases.

If $b = +\infty$, then $\operatorname{Re}\mu(\alpha) < 0$ provided that $\alpha > -\dfrac{c\|M\|}{\omega_1}$ (see above). Therefore, if the branch $\mu(\alpha)$ does not intersect the imaginary axis, then it is completely located in the left-hand half-plane. According to Lemma 5.2, such intersections are possible only for $\alpha < 1$; therefore, $\operatorname{Re}\mu(\alpha) < 0$ for $\alpha = 1$.

If $b < \infty$ and the function $\mu(\alpha)$ is not defined for $\alpha \geq b$, then $|\mu(\alpha)| \to \infty$ as $\alpha \to b - 0$. However, from estimates (5.3)-(5.4), we see that an unbounded growth of $|\mu(\alpha)|$ for bounded α is possible only under assumption that $\operatorname{Re}\mu(\alpha) \to -\infty$. Then, obviously, there exists α from $(1, b)$ such that $\operatorname{Re}\mu(\alpha) < 0$. Therefore, $\mu(1)$ is located in the left-hand half-plane again. Thus, it is proved that the spectrum of the operator $-\Gamma + M$ is located in the left-hand half-plane, i.e.,

$$s(-\Gamma + M) \leq 0.$$

Since $-\Gamma + M$ is an operator with a compact resolvent, it follows that its spectrum cannot have limit points distinct from ∞. On the other hand, its spectrum is completely located in the sector defined by (5.3)-(5.4) for $\alpha = 1$. Therefore, there is no $\{\lambda_i\} \subset \sigma(-\Gamma + M)$ such that $\operatorname{Re}\lambda_i \to 0$. This means that

$$s(-\Gamma + M) < 0.$$

Since $e^{(-\Gamma_C + M_C)t}$ is an analytic semigroup, it follows that it is continuous with respect to the norm for positive t; since the resolvent of its generator is compact, it follows that this semigroup is compact for positive t (see [3, Th. 2.4.29 and the next diagram]). There is no essential spectrum for positive t. Therefore,

$$\omega_0(e^{(-\Gamma+M)t}) = s(-\Gamma + M) < 0,$$

i.e., $e^{(-\Gamma+M)t}$ is uniformly exponentially stable. □

6. Example: Infinite Matrices

Consider the sequence space l_1 with the cone

$$K = \{\{x_k\}_{k\in\mathbb{N}} \in l_1 : \forall(k \in \mathbb{N})[x_k \geq 0]\}.$$

Let $\{\gamma_k\}$ be a sequence of positive numbers such that $\lim\limits_{k\to\infty} \gamma_k = +\infty$. Then, due to [15, Th. 2], the following operators are well defined, the domain $D(\Gamma)$ is dense in l_1, and Γ^{-1} is compact:

$$\Gamma : D(\Gamma) \subset l_1 \to l_1,$$

$$\Gamma x = \begin{pmatrix} \gamma_1 & 0 & \dots & 0 & \dots \\ 0 & \gamma_2 & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \gamma_k & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \gamma_1 x_1 \\ \gamma_2 x_2 \\ \vdots \\ \gamma_i x_k \\ \vdots \end{pmatrix}, \text{ and } \Gamma^{-1} x = \begin{pmatrix} \frac{1}{\gamma_1} & 0 & \dots & 0 & \dots \\ 0 & \frac{1}{\gamma_2} & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \frac{1}{\gamma_k} & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \frac{x_1}{\gamma_1} \\ \frac{x_2}{\gamma_2} \\ \vdots \\ \frac{x_k}{\gamma_k} \\ \vdots \end{pmatrix}.$$

The operator $-\Gamma$ generates the following contraction semigroup, strongly continuous and positive in the sense of the cone K:

$$e^{-\Gamma t} = \begin{pmatrix} e^{-\gamma_1 t} & 0 & \dots & 0 & \dots \\ 0 & e^{-\gamma_2 t} & \dots & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & e^{-\gamma_k t} & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}$$

Let $r > 0$ and $s \neq 0$. Then

$$\|R(r+is,-\Gamma)\| = \|(-\Gamma-(r+is)I\| = \left\|\begin{pmatrix} \frac{1}{-\gamma_1-r-is} & 0 & \cdots & 0 & \cdots \\ 0 & \frac{1}{-\gamma_2-r-is} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \frac{1}{-\gamma_k-r-is} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}\right\|$$

$$= \frac{1}{|s|}\left\|\begin{pmatrix} \frac{|s|}{\gamma_1+r+is} & 0 & \cdots & 0 & \cdots \\ 0 & \frac{|s|}{\gamma_2+r+is} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \frac{|s|}{\gamma_k+r+is} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}\right\| \leq \frac{1}{|s|}\sup_{k\in\mathbb{N}}\left\{\frac{|s|}{|\gamma_k+r+is|}\right\} < \frac{1}{|s|};$$

by virtue of [3, Theorem II.4.6.d], this implies the analyticity of the semigroup $e^{-\Gamma t}$.

Thus, the operator Γ satisfies all conditions of Theorem 5.1 and the strongly continuous semigroup $e^{(-\Gamma+M)t}$ is stable provided that M is a positive bounded linear operator such that $\rho(\Gamma^{-1}M) < 1$. Note that

$$\Gamma^{-1}M = \begin{pmatrix} \frac{m_{11}}{\gamma_1} & \frac{m_{12}}{\gamma_1} & \cdots & \frac{m_{1q}}{\gamma_1} & \cdots \\ \frac{m_{21}}{\gamma_2} & \frac{m_{22}}{\gamma_2} & \cdots & \frac{m_{2q}}{\gamma_2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \\ \frac{m_{p1}}{\gamma_p} & \frac{m_{p2}}{\gamma_p} & \cdots & \frac{m_{pq}}{\gamma_p} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}$$

and the operator l_1-norm satisfies the relation

$$\rho(\Gamma^{-1}M) \leq \|\Gamma^{-1}M\| = \sup_{q\in\mathbb{N}}\sum_{p=1}^{\infty}\frac{m_{pq}}{\gamma_p}.$$

Thus, using Theorem 5.1, one can obtain the following fact.

Theorem 6.1. *Let the operator Γ be defined above and the operator $M : l_1 \to l_1$, $M = \{m_{pq}\}_{p,q\in\mathbb{N}}$, be K-positive (i.e., $m_{pq} \geq 0$) and bounded (i.e., $\sup\limits_{q\in\mathbb{N}}\sum\limits_{p\in\mathbb{N}} m_{pq} < +\infty$). Then the condition*

$$\sup_{q\in\mathbb{N}}\sum_{p=1}^{\infty}\frac{m_{pq}}{\gamma_p} < 1 \tag{6.1}$$

implies the stability of the semigroup $e^{(-\Gamma+M)t}$.

Using the localization of eigenvalues by means of Geršgorin circles (for the case of infinite matrices, see, e.g., [15]), one can obtain various sufficient stability conditions for $e^{(-\Gamma+M)t}$ expressed by inequalities

$$\forall p \in \mathbb{N} \qquad \frac{\sum\limits_{q\in\mathbb{N}} m_{pq}}{\gamma_p} < 1 \tag{6.2}$$

and

$$\forall q \in \mathbb{N} \qquad \frac{\sum\limits_{p\in\mathbb{N}} m_{pq}}{\gamma_q} < 1. \tag{6.3}$$

However, these conditions are not equivalent to each other: e.g., the matrices

$$\Gamma = \begin{pmatrix} 10 & 0 & 0 & \dots \\ 0 & 20 & 0 & \dots \\ 0 & 0 & 30 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 5 & 1 & 1 & 1 & \dots \\ 5 & 1 & 1 & 1 & \dots \\ 5 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

satisfy (6.1), but do not satisfy (6.2) or (6.3). For the case of operators acting in the space l_∞, a condition for the perturbation can be deduced from Theorem 5.1 in the same way; however, in that case it provides no advantages compared with the Geršgorin method.

7. Example: Perturbations of Boundary-Value Problems by Operators with Delays

7.1. The original operator. Let the space F be $C[0,1]$ and the operator L be defined by the differential expression

$$l(f) = f''$$

and the boundary-value conditions

$$f'(0) = 0, \quad f'(1) = 0, \tag{7.1}$$

i.e.,

$$L : D(L) = \{f \in C^2[0,1] : f'(0) = f'(1) = 0\} \subset F \to F,$$
$$L : f \mapsto f''.$$

The operator L has the eigenvalue $\lambda_0 = 0$ with the eigenvectors $f_0(t) \equiv c$ and the eigenvalues $\lambda_k = -\pi^2 k^2$, $k \in \mathbb{N}$, with the eigenvectors $f_k(t) = c \cos \pi k t$. Its spectrum contains nothing else.

7.2. The Green function and resolvent. To find the resolvent of the operator L for regular λ, i.e., the operator $(L - \lambda I)^{-1}$, construct the Green function of the boundary-value problem

$$\begin{cases} f'' - \lambda f = g, \\ U_1(f) := f'(0) = 0, \\ U_2(f) := f'(1) = 0. \end{cases} \tag{7.2}$$

For the homogeneous system, we know the fundamental system of solutions

$$f_1(t) = e^{\sqrt{\lambda} t}, \quad f_2(t) = e^{-\sqrt{\lambda} t}$$

and their derivatives

$$f_1'(t) = \sqrt{\lambda} e^{\sqrt{\lambda} t}, \quad f_2'(t) = -\sqrt{\lambda} e^{-\sqrt{\lambda} t}.$$

Following the proof of [12, Th 1, I.§3.3], construct the Green function. It is known that

$$G_\lambda(x, \xi) = \begin{cases} 0 \le x < \xi \le 1 : a_1(\xi) f_1(x) + a_2(\xi) f_2(x), \\ 0 \le \xi < x \le 1 : b_1(\xi) f_1(x) + b_2(\xi) f_2(x), \end{cases} \tag{7.3}$$

where a and b satisfy the conditions

$$\begin{cases} a_1(\xi) f_1(\xi) + a_2(\xi) f_2(\xi) - (b_1(\xi) f_1(\xi) + b_2(\xi) f_2(\xi)) = 0, \\ a_1(\xi) f_1'(\xi) + a_2(\xi) f_2'(\xi) - (b_1'(\xi) f_1(\xi) + b_2'(\xi) f_2(\xi)) = -1. \end{cases}$$

Introducing the notation

$$c_1 = b_1 - a_1, \quad c_2 = b_2 - a_2, \tag{7.4}$$

we obtain the system

$$\begin{cases} c_1(\xi) e^{\sqrt{\lambda} \xi} + c_2(\xi) e^{\sqrt{-\lambda} \xi} = 0, \\ c_1(\xi) \sqrt{\lambda} e^{\sqrt{\lambda} \xi} - \sqrt{\lambda} e^{-\sqrt{\lambda} \xi} = 1. \end{cases}$$

Let us solve it:

$$W(\xi)=\begin{vmatrix} e^{\sqrt{\lambda}\xi} & e^{\sqrt{-\lambda}\xi} \\ \sqrt{\lambda}e^{\sqrt{\lambda}\xi} & -\sqrt{\lambda}e^{\sqrt{-\lambda}\xi}\end{vmatrix} = -\sqrt{\lambda}e^{(\sqrt{\lambda}\xi-\sqrt{\lambda}\xi)}-\sqrt{\lambda}e^{(\sqrt{\lambda}\xi-\sqrt{\lambda}\xi)}=-2\sqrt{\lambda},$$

$$W_1(\xi)=\begin{vmatrix} 0 & e^{-\sqrt{\lambda}\xi} \\ 1 & -\sqrt{\lambda}e^{-\sqrt{\lambda}}\end{vmatrix}=-e^{-\sqrt{\lambda}\xi},\quad W_2(\xi)=\begin{vmatrix} e^{\sqrt{\lambda}\xi} & 0 \\ \sqrt{\lambda}e^{\sqrt{\lambda}} & 1\end{vmatrix}=e^{\sqrt{\lambda}\xi},$$

$$c_1(\xi)=\frac{W_1(\xi)}{W(\xi)}=\frac{-e^{-\sqrt{\lambda}\xi}}{-2\sqrt{\lambda}}=\frac{e^{-\sqrt{\lambda}\xi}}{2\sqrt{\lambda}},\quad c_2(\xi)=\frac{W_2(\xi)}{W(\xi)}=\frac{e^{\sqrt{\lambda}\xi}}{-2\sqrt{\lambda}}=-\frac{e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}.$$

The boundary-value conditions yield the system

$$\begin{cases} U_1(G_\lambda)=0, \\ U_2(G_\lambda)=0. \end{cases}$$

From this, taking into account (7.3) and (7.4) (see [12, I.§3.3]), we obtain the system

$$\begin{cases} b_1(\xi)\sqrt{\lambda}e^{\sqrt{\lambda}\cdot 0}-b_2(\xi)\sqrt{\lambda}e^{-\sqrt{\lambda}\cdot 0}=\frac{e^{-\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}\sqrt{\lambda}e^{\sqrt{\lambda}\cdot 0}+\frac{e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}\sqrt{\lambda}e^{-\sqrt{\lambda}\cdot 0}, \\ b_1(\xi)\sqrt{\lambda}e^{\sqrt{\lambda}\cdot 1}-b_2(\xi)\sqrt{\lambda}e^{-\sqrt{\lambda}\cdot 1}=0, \end{cases}$$

i.e.,

$$\begin{cases} b_1(\xi)\sqrt{\lambda}-b_2(\xi)\sqrt{\lambda}=\frac{e^{-\sqrt{\lambda}\xi}}{2}+\frac{e^{\sqrt{\lambda}\xi}}{2}, \\ b_1(\xi)e^{\sqrt{\lambda}}=b_2(\xi)e^{-\sqrt{\lambda}}, \end{cases}$$

i.e.,

$$\begin{cases} \sqrt{\lambda}(b_1(\xi)-b_2(\xi))=\frac{e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi}}{2}, \\ b_1(\xi)=b_2(\xi)e^{-2\sqrt{\lambda}}, \end{cases}$$

i.e.,

$$\begin{cases} \sqrt{\lambda}b_2(\xi)(e^{-2\sqrt{\lambda}}-1)=\frac{e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi}}{2}, \\ b_1(\xi)=b_2(\xi)e^{-2\sqrt{\lambda}}. \end{cases}$$

Note that $\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)=0\Rightarrow\{-\pi^2k^2 : k\in\{0\}\cup\mathbb{N}\}$. Then

$$\begin{cases} b_2(\xi)=\frac{e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)} \\ b_1(\xi)=\frac{(e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi})e^{-2\sqrt{\lambda}}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)} \end{cases}.$$

Then

$$a_1(\xi)=b_1(\xi)-c_1(\xi)=\frac{(e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi})e^{-2\sqrt{\lambda}}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}-\frac{e^{-\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}$$

and

$$a_2(\xi)=b_2(\xi)-c_2(\xi)=\frac{e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}+\frac{e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}.$$

Finally, we have

$$G_\lambda(x,\xi)=\begin{cases} 0\le x<\xi\le 1 : G_{1,\lambda}(x,\xi) \\ 0\le \xi<x\le 1 : G_{2,\lambda}(x,\xi) \end{cases},$$

where

$$G_{1,\lambda}(x,\xi)=\left(\frac{(e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi})e^{-2\sqrt{\lambda}}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}-\frac{e^{-\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}\right)e^{\sqrt{\lambda}x}+\left(\frac{e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}+\frac{e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}}\right)e^{-\sqrt{\lambda}x}$$

and

$$G_{2,\lambda}(x,\xi)=\frac{(e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi})e^{-2\sqrt{\lambda}}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}e^{\sqrt{\lambda}x}+\frac{e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi}}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}e^{-\sqrt{\lambda}x}.$$

This can be simplified as follows:

$$G_{1,\lambda}(x,\xi) = \frac{1}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}\Big(e^{-\sqrt{\lambda}\xi-2\sqrt{\lambda}+\sqrt{\lambda}x}+e^{\sqrt{\lambda}\xi-2\sqrt{\lambda}+\sqrt{\lambda}x}-e^{-2\sqrt{\lambda}-\sqrt{\lambda}\xi+\sqrt{\lambda}x}+e^{-\sqrt{\lambda}\xi+\sqrt{\lambda}x}$$
$$+e^{-\sqrt{\lambda}\xi-\sqrt{\lambda}x}+e^{\sqrt{\lambda}\xi-\sqrt{\lambda}x}+e^{-2\sqrt{\lambda}+\sqrt{\lambda}\xi-\sqrt{\lambda}x}-e^{\sqrt{\lambda}\xi-\sqrt{\lambda}x}\Big)$$
$$= \frac{1}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}\Big(e^{\sqrt{\lambda}\xi-2\sqrt{\lambda}+\sqrt{\lambda}x}+e^{-\sqrt{\lambda}\xi+\sqrt{\lambda}x}+e^{-\sqrt{\lambda}\xi-\sqrt{\lambda}x}+e^{-2\sqrt{\lambda}+\sqrt{\lambda}\xi-\sqrt{\lambda}x}\Big)$$
$$= \frac{(e^{-2\sqrt{\lambda}}e^{\sqrt{\lambda}\xi}+e^{-\sqrt{\lambda}\xi})(e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x})}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}$$

and

$$G_{2,\lambda}(x,\xi) = \frac{(e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi})(e^{-2\sqrt{\lambda}}e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x})}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}.$$

Once we obtain the Green function, we can express the resolvent of the operator L:

$$R(\lambda, L)(f) = \int\limits_0^1 G_\lambda(x,\xi)f(\xi)d\xi,$$

where $f \in C[0,1]$.

7.3. The sectoriality of the operator. To show that L is a sectorial operator (see [3, II.4.a]), it suffices to prove that there exists θ from $\left(\frac{\pi}{2},\pi\right)$ such that the sector $\Sigma_\theta = \{\lambda \in \mathbb{C} : |\arg\lambda| < \theta\} \setminus \{0\}$ consists only of regular points and there exists M such that the following inequality holds for any such point:

$$||R(\lambda, L)|| \le \frac{M}{|\lambda|}. \tag{7.5}$$

Fix θ from $\left(\frac{\pi}{2},\pi\right)$. Apart from the point 0, the spectrum of L is located on the real axis in the left-hand half-plane. Therefore, it is not located in Σ_θ. Now, to prove estimate (7.5), we note that

$$||R(\lambda, L)|| = \sup_{f\in C[0,1],||f||=1} ||R(\lambda,L)f|| = \sup_{f\in C[0,1],||f||=1}\max_{x\in[0,1]}\left|\int\limits_0^1 G_\lambda(x,\xi)f(\xi)d\xi\right|$$
$$\le \sup_{f\in C[0,1],||f||=1}\max_{x\in[0,1]}\int\limits_0^1 |G_\lambda(x,\xi)f(\xi)|d\xi \le \sup_{f\in C[0,1],||f||=1}\max_{x\in[0,1]}\int\limits_0^1 |G_\lambda(x,\xi)|\max_{y\in[0,1]}|f(y)|d\xi$$
$$= \sup_{f\in C[0,1],||f||=1}\max_{x\in[0,1]}\left(\int\limits_0^1 |G_\lambda(x,\xi)|d\xi\, ||f||\right) = \max_{x\in[0,1]}\int\limits_0^1 |G_\lambda(x,\xi)|d\xi$$
$$= \max_{x\in[0,1]}\left(\int\limits_0^x |G_{2,\lambda}(x,\xi)|d\xi + \int\limits_x^1 |G_{1,\lambda}(x,\xi)|d\xi\right)$$
$$= \max_{x\in[0,1]}\left(\int\limits_0^x\left|\frac{(e^{-\sqrt{\lambda}\xi}+e^{\sqrt{\lambda}\xi})(e^{-2\sqrt{\lambda}}e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x})}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}\right|d\xi\right.$$
$$\left.+\int\limits_x^1\left|\frac{(e^{-2\sqrt{\lambda}}e^{\sqrt{\lambda}\xi}+e^{-\sqrt{\lambda}\xi})(e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x})}{2\sqrt{\lambda}(e^{-2\sqrt{\lambda}}-1)}\right|d\xi\right)$$

$$\leq \frac{1}{2|\sqrt{\lambda}||e^{-2\sqrt{\lambda}}-1|}\max_{x\in[0,1]}\Bigg(\left|e^{-2\sqrt{\lambda}}e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x}\right|\int\limits_0^x\left(\left|e^{-\sqrt{\lambda}\xi}\right|+\left|e^{\sqrt{\lambda}\xi}\right|\right)d\xi$$

$$+\left|e^{\sqrt{\lambda}x}+e^{-\sqrt{\lambda}x}\right|\int\limits_x^1\left(\left|e^{-\sqrt{\lambda}\xi}\right|+\left|e^{-2\sqrt{\lambda}}\right|\left|e^{\sqrt{\lambda}\xi}\right|\right)d\xi\Bigg)$$

$$\leq \frac{1}{2|\sqrt{\lambda}||e^{-2\sqrt{\lambda}}-1|}\max_{x\in[0,1]}\Bigg(\left(e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}x}\right)\left(\frac{e^{-\mathrm{Re}\sqrt{\lambda}x}-1}{-\mathrm{Re}\sqrt{\lambda}}+\frac{e^{\mathrm{Re}\sqrt{\lambda}x}-1}{\mathrm{Re}\sqrt{\lambda}}\right)$$

$$+\left(e^{\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}x}\right)\left(\frac{e^{-\mathrm{Re}\sqrt{\lambda}}-e^{-\mathrm{Re}\sqrt{\lambda}x}}{-\mathrm{Re}\sqrt{\lambda}}+e^{-2\mathrm{Re}\sqrt{\lambda}}\frac{e^{\mathrm{Re}\sqrt{\lambda}}-e^{\mathrm{Re}\sqrt{\lambda}x}}{\mathrm{Re}\sqrt{\lambda}}\right)\Bigg)=\frac{1}{2|\sqrt{\lambda}||e^{-2\sqrt{\lambda}}-1|}$$

$$\times\max_{x\in[0,1]}\Bigg(\frac{1}{-\mathrm{Re}\sqrt{\lambda}}\left(e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}e^{-\mathrm{Re}\sqrt{\lambda}x}-e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}x}e^{-\mathrm{Re}\sqrt{\lambda}x}-e^{-\mathrm{Re}\sqrt{\lambda}x}\right)$$

$$+\frac{1}{\mathrm{Re}\sqrt{\lambda}}\left(e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}e^{\mathrm{Re}\sqrt{\lambda}x}-e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}x}e^{\mathrm{Re}\sqrt{\lambda}x}-e^{-\mathrm{Re}\sqrt{\lambda}x}\right)$$

$$+\frac{1}{-\mathrm{Re}\sqrt{\lambda}}\left(e^{\mathrm{Re}\sqrt{\lambda}x}e^{-\mathrm{Re}\sqrt{\lambda}}-e^{\mathrm{Re}\sqrt{\lambda}x}e^{-\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}x}e^{-\mathrm{Re}\sqrt{\lambda}}-e^{-\mathrm{Re}\sqrt{\lambda}x}e^{-\mathrm{Re}\sqrt{\lambda}x}\right)$$

$$+\frac{1}{\mathrm{Re}\sqrt{\lambda}}\Big(e^{\mathrm{Re}\sqrt{\lambda}x}e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}}-e^{\mathrm{Re}\sqrt{\lambda}x}e^{-2\mathrm{Re}\sqrt{\lambda}}e^{-\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}x}e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}}$$

$$-e^{-\mathrm{Re}\sqrt{\lambda}x}e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}\Big)\Bigg)=\frac{1}{2|\sqrt{\lambda}||e^{-2\sqrt{\lambda}}-1||\mathrm{Re}\sqrt{\lambda}|}\max_{x\in[0,1]}\Big(-e^{-2\mathrm{Re}\sqrt{\lambda}}+e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}$$

$$-e^{-2\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}x}+e^{-2\mathrm{Re}\sqrt{\lambda}}e^{2\mathrm{Re}\sqrt{\lambda}x}-e^{-2\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}+1-e^{-\mathrm{Re}\sqrt{\lambda}x}-e^{-\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}+1$$

$$-e^{-\mathrm{Re}\sqrt{\lambda}}e^{-\mathrm{Re}\sqrt{\lambda}x}+e^{-2\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}}e^{\mathrm{Re}\sqrt{\lambda}x}-e^{-2\mathrm{Re}\sqrt{\lambda}}e^{2\mathrm{Re}\sqrt{\lambda}x}+e^{-\mathrm{Re}\sqrt{\lambda}}e^{-\mathrm{Re}\sqrt{\lambda}x}-e^{-2\mathrm{Re}\sqrt{\lambda}}\Big)$$

$$=\frac{|e^{-2\mathrm{Re}\sqrt{\lambda}}-1|}{|\sqrt{\lambda}||\mathrm{Re}\sqrt{\lambda}||e^{-2\sqrt{\lambda}}-1|}. \tag{7.6}$$

Since $|\arg\lambda|<\theta<\pi$, it follows that the root located in the right-hand half-plane can be chosen as $\sqrt{\lambda}$; then, without loss of generality, one can assume that $0\leq|\arg\sqrt{\lambda}|<\dfrac{\theta}{2}<\dfrac{\pi}{2}$. Then $|\mathrm{Re}\sqrt{\lambda}|=|\sqrt{\lambda}|\cos|\arg\sqrt{\lambda}|>|\sqrt{\lambda}|\cos\dfrac{\theta}{2}$.

On the other hand, for any positive a and $b>0$ and any $|\varphi|$ such that $|\varphi|<\dfrac{\pi}{2}$, the following relation is valid:

$$|a(\cos\varphi+i\sin\varphi)-b|=\sqrt{(a\cos\varphi-b)^2+a^2\sin^2\varphi}$$

$$=\sqrt{a^2\cos^2\varphi-2a\cos\varphi\cdot b+b^2+a^2\sin^2\varphi}=\sqrt{a^2(\cos^2\varphi+\sin^2\varphi)-2ab\cos\varphi+b^2}$$

$$=\sqrt{a^2-2ab\cos\varphi+b^2}\geq\sqrt{a^2-2ab+b^2}=\sqrt{(a-b)^2}=a-b.$$

In our case, we have $a=e^{-2\mathrm{Re}\sqrt{\lambda}}, b=1$, and $\varphi=\arg\sqrt{\lambda}$. Therefore, it follows from (7.6) that

$$R(\lambda,L)\leq\frac{1}{|\lambda|\cos\frac{\theta}{2}}.$$

This completes the proof of the sectoriality of L.

7.4. Invertible operators and semigroups. In [3, II.2.12], it is shown that the operator L generates a strongly continuous semigroup in the space $F = C[0,1]$ defined as follows:

$$(e^{Lt} f)(s) = \int_0^1 k_t(s,r) f(r) dr,$$

$$k_t(s,r) = 1 + 2 \sum_{n \in \mathbb{N}} e^{-\pi^2 n^2 t} \cos(\pi n s) \cos(\pi n r),$$

where k_t are positive functions defined on $[0,1] \times [0,1]$.

Since $0 \in \sigma(L)$, it follows that the operator L is irreversible. Construct a scaled semigroup selecting a positive a and introducing the notation

$$\Gamma = -L + aI.$$

Then, due to [3, II.2.2], we have

$$\sigma(-\Gamma) = \sigma(L - aI) = \{-\pi^2 k^2 - a : k \in \{0\} \cup \mathbb{N}\}$$

and

$$e^{-\Gamma} = e^{-at} e^{Lt}. \tag{7.7}$$

Since L is a sectorial operator, we see that it generates an analytic semigroup (see [3, Theorem II.4.6]); hence, $-\Gamma = L - aI$ generates an analytic semigroup as well because $-aI$ is a bounded operator (see [3, Proposition III.1.12]). Since

$$R(\lambda, \Gamma) = (\Gamma - \lambda I)^{-1} = (-(-\Gamma - (-\lambda) I))^{-1} = -(-\Gamma - (-\lambda) I)^{-1} = -R(-\Gamma, -\lambda),$$

it follows that

$$\sigma(\Gamma) = -\sigma(-\Gamma) = \{\pi^2 k^2 + a : k \in \{0\} \cup \mathbb{N}\}$$

with the same eigensubspaces for the corresponding eigenvalues.

We see that 0 is a regular point of the operator Γ. Therefore, it is invertible. Due to [6, Th. 6.15, III.6.3], we have the extended spectrum (the only possible distinctions between it and the spectrum of the operator are the points 0 and ∞)

$$\widetilde{\sigma}(\Gamma^{-1}) = \{0\} \cup \left\{ \frac{1}{\pi^2 k^2 + a} : k \in \{0\} \cup \mathbb{N} \right\}$$

and obtain the spectral radius

$$\sigma(\Gamma^{-1}) \ni \frac{1}{a} = \rho(\Gamma^{-1}).$$

Note that

$$(\Gamma^{-1} f)(x) = ((-L + aI)^{-1} f)(x) = ((-(L - aI))^{-1} f)(x)$$

$$(-(L - aI)^{-1} f)(x) = -\int_0^1 G_a(x,\xi) f(\xi) d\xi. \tag{7.8}$$

Then the operator Γ^{-1} is compact (see [7, IV.§6.1.4]). Hence, the resolvent Γ and the operator $-\Gamma$ are compact as well. Since $e^{-\Gamma t}$ is an analytic semigroup, it follows that it is continuous with respect to the norm for positive $t > 0$. Then, due to the compactness of the resolvent of the generator, the semigroup itself is compact as well (see [3, Theorem II.4.29]). Hence, it has no essential spectrum and $\omega_0(e^{-\Gamma t}) = s(-\Gamma) = \sup \operatorname{Re} \sigma(-\Gamma) = -a < 0$, i.e., the semigroup is stable.

Consider the cone

$$K = \{f \in C[0,1] : \forall (x \in [0,1])[f(x) \geq 0]\}.$$

If only operators over real-valued functions are considered, then the semigroup $e^{-\Gamma t}$ is positive in the sense of K because the kernels k_t are positive (this implies the positivity of e^{Lt}) and relation (7.7) holds.

7.5. Examples of perturbations with delays. Thus, all conditions of Theorem 5.1 are fulfilled for the operator $-\Gamma$. It implies the following corollary.

Theorem 7.1. *Let $M : F \to F$ be a bounded linear operator positive with respect to the cone K and such that $\rho(\Gamma^{-1}M) < 1$. Then the semigroup $e^{(-\Gamma+M)t}$ is uniformly exponentially stable.*

Example 7.1. Fix p from $(0, 1)$. Take a function q_1 from K and consider the operator

$$(M_1 f)(t) = \begin{cases} t \in [0, p] : q_1(t)f(0) \\ t \in [p, 1] : q_1(t)f(t-p) \end{cases}.$$

The operator M_1 is positive and bounded. To find out conditions of the uniformly exponential stability of the semigroup $e^{(-\Gamma+M_1)t}$, note that

$$\begin{aligned}\rho(\Gamma^{-1}M_1) \le \|\Gamma^{-1}M_1\| &= \sup_{f\in C[0,1],\|f\|=1} \max_{x\in[0,1]} |(\Gamma^{-1}M_1 f)(x)| \\ &= \sup_{f\in C[0,1],\|f\|=1} \max_{x\in[0,1]} \left| -\int_0^1 G_a(x,\xi)M_1 f(\xi)d\xi \right| \\ &\le \sup_{f\in C[0,1],\|f\|=1} \max_{x\in[0,1]} \left(\int_0^p |G_a(x,\xi)|\,|q_1(\xi)|\,|f(0)|d\xi + \int_p^1 |G_a(x,\xi)|\,|q_1(\xi)|\,|f(\xi-p)|d\xi \right) \\ &\le \sup_{f\in C[0,1],\|f\|=1} \max_{x\in[0,1]} \int_0^1 |G_a(x,\xi)|\,|q_1(\xi)|d\xi\|f\| = \max_{x\in[0,1]} \int_0^1 |G_a(x,\xi)|q_1(\xi)d\xi.\end{aligned}$$

Thus, $e^{(-\Gamma+M_1)t}$ is stable if

$$\max_{x\in[0,1]} \int_0^1 |G_a(x,\xi)|q_1(\xi)d\xi < 1.$$

If

$$\left[\int_0^p |G_a(x,\xi)|q_1(\xi)d\xi \int_p^1 |G_a(x,\xi)|q_1(\xi)d\xi \neq 0 \right] \tag{7.9}$$

for all x from $[1-p, 1]$, then the estimate can be improved.

Let $\nu(t)$ be continuous on $[0, 1]$ and let $\nu(t) \neq 0$ for any t from $[0, 1]$. Introduce the following norm in $C[0, 1]$:

$$\|x\|_\nu = \|\nu x\|.$$

It is equivalent to the standard one:

$$\|x\| = \left\|\frac{\nu}{\nu}x\right\| = \max_{t\in[0,1]} \left|\frac{\nu(t)}{\nu(t)}x(t)\right| \le \max_{t\in[0,1]} \left|\frac{1}{\nu(t)}\right| \max_{t\in[0,1]} |\nu(t)x(t)| = \left\|\frac{1}{\nu}\right\| \|x\|_\nu$$

and

$$\|x\|_\nu = \|x\nu\| = \max_{t\in[0,1]} |x(t)\nu(t)| \le \max_{t\in[0,1]} |x(t)| \max_{t\in[0,1]} |\nu(t)| = \|x\|\|\nu\|.$$

Then, similarly to the case of the standard norm, we have the estimate

$$\rho(\Gamma^{-1}M_1) \le \|\Gamma^{-1}M_1\|_\nu = \sup_{f\in C[0,1],\|f\|_\nu=1} \max_{x\in[0,1]} \left| -\int_0^1 G_a(x,\xi)M_1 f(\xi)d\xi\nu(x) \right|$$

$$\leq \sup_{f\in C[0,1],\|f\|_\nu=1} \max_{x\in[0,1]} \left(\int\limits_0^p |G_a(x,\xi)|q_1(\xi) \left| f(0)\frac{\nu(0)}{\nu(0)} \right| d\xi |\nu(x)| \right.$$

$$\left. + \int\limits_p^1 |G_a(x,\xi)|q_1(\xi) \left| f(\xi-p)\frac{\nu(\xi-p)}{\nu(\xi-p)} \right| d\xi |\nu(x)| \right)$$

$$\leq \sup_{f\in C[0,1],\|f\|_\nu=1} \max_{x\in[0,1]} \left(\int\limits_0^p |G_a(x,\xi)|q_1(\xi) d\xi \|f\|_\nu \left| \frac{\nu(x)}{\nu(0)} \right| \right.$$

$$\left. + \int\limits_p^1 |G_a(x,\xi)|q_1(\xi) d\xi \|f\|_\nu \max_{\xi\in[p,1]} \left| \frac{\nu(x)}{\nu(\xi-p)} \right| \right)$$

$$= \max_{x\in[0,1]} \left(\int\limits_0^p |G_a(x,\xi)|q_1(\xi) d\xi \left| \frac{\nu(x)}{\nu(0)} \right| + \int\limits_p^1 |G_a(x,\xi)|q_1(\xi) d\xi \max_{\xi\in[p,1]} \left| \frac{\nu(x)}{\nu(\xi-p)} \right| \right). \tag{7.10}$$

Assign

$$\nu(x) = \begin{cases} 1, & x \leq 1-p, \\ \min\left(1, \frac{\int_0^p |G_a(1-p,\xi)|q_1(\xi)d\xi}{\int_0^p |G_a(x,\xi)|q_1(\xi)d\xi}\right) \min\left(1, \frac{\int_p^1 |G_a(1-p,\xi)|q_1(\xi)d\xi}{\int_p^1 |G_a(x,\xi)|q_1(\xi)d\xi}\right), & 1-p < x. \end{cases}$$

Due to (7.9), the function $\nu(x)$ is defined, continuous, and positive in $[0,1]$. If $x > 1-p$, then

$$\int\limits_0^p |G_a(x,\xi)|q_1(\xi) d\xi \left| \frac{\nu(x)}{\nu(0)} \right| + \int\limits_p^1 |G_a(x,\xi)|q_1(\xi) d\xi \max_{\xi\in[p,1]} \left| \frac{\nu(x)}{\nu(\xi-p)} \right|$$

$$\leq \int\limits_0^p |G_a(1-p,\xi)|q_1(\xi) d\xi \nu(1-p) + \int\limits_p^1 |G_a(1-p,\xi)|q_1(\xi) d\xi \nu(1-p).$$

Hence, the maximum with respect to x in (7.10) is attained if $x \in [0, 1-p]$ and the inequality

$$\max_{x\in[0,1-p]} \int\limits_0^1 |G_a(x,\xi)|q_1(\xi) d\xi < 1$$

is a stability condition provided that (7.9) is fulfilled.

Example 7.2. Let $q_2 : [0,1] \times [0,1] \to [0,1]$ be a continuous function such that $q_2(x,s) \geq 0$ for all x and s from $[0,1]$. Consider the operator

$$(M_2 f)(s) = \int\limits_0^p q_2(t,s) f(0) ds + \int\limits_p^1 q_2(t,s) f(s-p) ds.$$

In the same way, we have

$$\rho(\Gamma^{-1}M_1) \le \sup_{f\in C[0,1],\|f\|=1} \max_{x\in[0,1]} \left| -\int_0^1 G_a(x,\xi) \left(\int_0^p q_2(\xi,s)f(0)ds + \int_p^1 q_2(\xi,s)f(s-p)ds \right) d\xi \right|$$

$$\le \max_{x\in[0,1]} \int_0^1 |G_a(x,\xi)| \int_0^1 q_2(\xi,s)ds\, d\xi.$$

Then we have the following stability condition for the semigroup $e^{(-\Gamma+M_2)t}$:

$$\max_{x\in[0,1]} \int_0^1 |G_a(x,\xi)| \int_0^1 q_2(\xi,s)ds\, d\xi < 1.$$

Similarly to Example 7.1, if

$$\int_0^1 \int_0^p |G_a(x,\xi)| q_2(\xi,s)ds\, d\xi \int_0^1 \int_p^1 |G_a(x,\xi)| q_2(\xi,s)ds\, d\xi \ne 0$$

for all x from $[1-p,1]$, then the following estimate is valid:

$$\rho(\Gamma^{-1}M_2) \le \|\Gamma^{-1}M_2\|_\nu$$

$$\le \max_{x\in[0,1]} \left(\int_0^1 \int_0^p |G_a(x,\xi)| q_2(\xi,s)ds\, d\xi \left| \frac{\nu(x)}{\nu(0)} \right| + \int_0^1 \int_p^1 |G_a(x,\xi)| q_2(\xi,s)ds\, d\xi \max_{s\in[p,1]} \left| \frac{\nu(x)}{\nu(s-p)} \right| \right).$$

Then, assigning

$$\nu(x) = \begin{cases} 1, & x \le 1-p, \\ \min\left(1, \dfrac{\int_0^1\int_0^p |G_a(1-p,\xi)| q_2(\xi,s)ds\, d\xi}{\int_0^1\int_0^p |G_a(x,\xi)| q_2(\xi,s)ds\, d\xi}\right) \min\left(1, \dfrac{\int_0^1\int_p^1 |G_a(1-p,\xi)| q_2(\xi,s)ds\, d\xi}{\int_0^1\int_p^1 |G_a(x,\xi)| q_2(\xi,s)ds\, d\xi}\right), & 1-p < x, \end{cases}$$

we obtain the following improved stability condition:

$$\max_{x\in[0,1-p]} \int_0^1 |G_a(x,\xi)| \int_0^1 q_2(\xi,s)ds\, d\xi < 1.$$

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