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GLOBAL ASYMPTOTIC STABILITY OF NONCONVEX SWEEPING PROCESSES

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Abstract. Building upon the technique that we developed earlier for perturbed sweeping processes with convex moving constraints and monotone vector fields (Kamenskii et al, Nonlinear Anal. Hybrid Syst. 30, 2018), the present paper establishes the conditions for global asymptotic stability of global and periodic solutions to perturbed sweeping processes with prox-regular moving constraints. Our conclusion can be formulated as follows: closer the constraint to a convex one, weaker monotonicity is required to keep the sweeping process globally asymptotically stable. We explain why the proposed technique is not capable to prove global asymptotic stability of a periodic regime in a crowd motion model (Cao-Mordukhovich, DCDS-B 22, 2017). We introduce and analyze a toy model which clarifies the extent of applicability of our result.

1. Introduction.

Let $t \mapsto C(t)$ be a set valued map which takes nonempty closed values and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Then the corresponding perturbed Moreau sweeping process is given as

$$-\dot{x} \in N\left(C(t), x\right) + f(t, x) \qquad (1)$$

where $N(C(t), \cdot)$ is the proximal normal cone to the set $C(t)$, given by

$$N(C, x) = \{v \in \mathbb{R}^n : x \in \text{proj}(x + \alpha v, C) \text{ for some } \alpha > 0\}$$

and $\text{proj}(x, C)$ is the set of points of C closest to the point x.

We say an absolutely continuous function x is a solution of the sweeping process (1) on an interval $I \subset \mathbb{R}$ if $x(t) \in C(t)$ for each t and $\dot{x}(t)$ satisfy (1) for a.e. $t \in I$.

Due to challenges from crowd motion modeling (Maury-Venel [20]), the existence and uniqueness of a solution to nonconvex sweeping processes have being intensively studied. The main problem of weakening the convexity of the set is the lack of continuity of the map $x \mapsto \text{proj}(x, C)$ in general. Therefore, the concept of prox-regularity came to the study of sweeping processes. A set $C \subset \mathbb{R}^n$ is called η-prox-regular if, for any $x \in C$ and any $v \in N(C, x)$ such that $\|v\| < 1$, one has $x = \text{proj}(x+\eta v, C)$. We note that, for η-prox-regular sets, the proximal normal cone coincides (see Edmond-Thibault [12], Rockafellar-Wets [24]) with both the *limiting*

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normal cone (also known as *Mordukhovich normal cone*, see [12]) and *Clarke normal cone*.

There has been a significant interest in developing the qualitative theory and control methods for sweeping processes with prox-regular constraints lately. Colombo-Goncharov [9], Benabdellah [3], Colombo and Monteiro Marques [10], and Thibault [27] studied the existence and uniqueness of solutions to non-perturbed sweeping processes with nonconvex prox-regular sets. Existence and uniqueness for perturbed sweeping processes is considered in Edmond-Thibault [11], [12]. A sweeping process with prox-regular set values appeared in the context of crowd motion modeling in Maury-Venel [20] along with numerical simulations. Cao-Mordukhovich [6] illustrate their result for nonconvex sweeping process using crowd motion model of traffic flow in a corridor. Edmond-Thibault [12], Cao-Mordukhovich [7] studied optimal control problems related to a nonconvex perturbed sweeping process. Optimal control problem of convex sweeping process which is coupled with a differential equation was studied in Adam-Outrata [1] and the possibility of weakening the convexity to prox-regularity is mentioned there.

The problem of the existence of periodic solutions in sweeping processes with convex constraint was of interest lately, see e.g. Krejci [16, Theorem 3.14], Castaing and Monteiro Marques [8, Theorem 5.3], Kunze [17], Kamenskii-Makarenkov [15], Kamenskii et al [14], and references therein. When the sweeping process comes as a model of an elastoplastic material (see e.g. Bastein et al [2]), the periodically changing constraint corresponds to the cyclic loading applied to the material (see Frederick-Armstrong [13], Polizzotto [23]). Much less is known for sweeping processes with nonconvex constraints (often termed *nonconvex sweeping processes*).

In this paper we investigate stability of both arbitrary global solution and a periodic solution of the sweeping process (1) with prox-regular set-valued function $C(t)$. The existence of globally exponentially stable global and periodic solutions to (1) when $C(t)$ is convex-valued has been recently established in Kamenskii et al [14]. The central setting of [14] is the strong monotonicity of f in the sense that

$$\langle f(t,x_1) - f(t,x_2), x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|^2, \quad \text{for all } t \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n, \qquad (2)$$

for some fixed $\alpha > 0$. A similar framework has been earlier used by Heemels-Brogliato [5], Brogliato [4] and Leine-van de Wouw [18] to prove incremental stability of the sweeping process (1) with time-independent convex constraint. An important breakthrough in establishing asymptotic stability of sweeping process (1) with a prox-regular constraint $C(t)$ has been made in Tanwani et al [26] for $f(t,x) = -Ax$ under the assumption that $\dot{x} = Ax$ admits a quadratic Lyapunov function. Our assumption (2) can be viewed as an analogue of the assumption of [26]. Indeed, the assumption (2) implies that the differential equation $-\dot{x} = f(t,x)$ admits a Lyapunov function $V(x) = \|x\|^2$ with $\dot{V}(x) \leq -2\alpha V(x)$.

The paper is organized as follows. The next section is devoted to the proof of the main result (Theorem 2.4), which gives conditions for the global asymptotic stability of a periodic solution to (1). The structure of the proof is motivated by the method of our paper [14]. Indeed, the existence of a global solution to (1) is justified by following the lines of the proof of Theorem 2.1 in [14] since the proof is independent of the convexity of the set (the proof of Theorem 2.2 is still given in Appendix for completeness). At the same time, additional assumptions, compared to [14] are still required. First of all, in order to use the hypomonotonicity of the proximal normal cone, we need $f(\cdot, x)$ to be globally bounded for each $x \in \bigcup_{t\in\mathbb{R}} C(t)$, additionally to the assumptions of Theorem 2.2 in [14]. Furthermore, to obtain contraction of solutions to sweeping process (1), a lower bound of constant α in (2) depending on prox-regularity constant of the set $C(t)$ is required (Theorem 2.3).

Section 3 is devoted to a toy model that illustrate the main result. Though global stability of the sweeping process of crowd motion model of Maury-Venel [20] has been the main driving force behind this paper, it still remains an open question as we discuss in the Appendix.

2. **The main result.** Let $C : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a nonempty closed η-prox-regular set-valued function with Lipschitz continuity

$$d_H(C(t_1), C(t_2)) \leq L_C|t_1 - t_2|, \quad \text{for all } t_1, t_2 \in \mathbb{R}, \text{ and for some } L_C \geq 0, \tag{3}$$

where $d_H(C_1, C_2)$ is the Hausdorff distance between two closed sets $C_1, C_2 \subset \mathbb{R}^n$ given by

$$d_H(C_1, C_2) = \max\left\{\sup_{x\in C_2} \operatorname{dist}(x, C_1), \sup_{x\in C_1} \operatorname{dist}(x, C_2)\right\} \tag{4}$$

with $\operatorname{dist}(x, C) = \inf\{|x - c| : c \in C\}$. And let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be such that for some $L_f > 0$

$$\|f(t_1, x_1) - f(t_2, x_2)\| \leq L_f\|t_1 - t_2\| + L_f\|x_1 - x_2\|, \tag{5}$$

for all $t_1, t_2 \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n$.

Here we will be using the hypomonotonicity of the proximal normal cone to η-prox-regular sets. A set-valued mapping $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called *hypomonotone* on $O \subset \mathbb{R}^n$ (Rockafellar-Wets [24, §12.28]), if there exists $\sigma > 0$ such that the mapping $\Phi + \sigma I$ is monotone on $\mathbb{R}^n$, i.e.

$$\langle v - v', x - x'\rangle \geq -\sigma\|x - x'\|, \quad v \in \Phi(x),\ v' \in \Phi(x'),\ x, x' \in O,$$

see also Mordukhovich [21, §5.1.1]. Define the truncated proximal normal cone $N^\eta(C, x)$ as

$$N^\eta(C, x) = \begin{cases} N(C, x) \cap B_\eta(0), & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

As established in Poliquin et al [22], if $C \subset \mathbb{R}^n$ is η-prox-regular, then the truncated mapping $x \rightrightarrows N^\eta(C, x)$ is hypomonotone on C and, therefore,

$$\langle v - v', x - x'\rangle \geq -\|x - x'\|^2 \tag{6}$$

for $v \in N(C, x), v' \in N(C, x')$ such that $\|v\|, \|v'\| \leq \eta$.

We will be using the following version of Gronwall-Bellman lemma Trubnikov-Perov [28, Lemma 1.1.1.5] (see also Kamenskii et al [14, lemma 6.1]) in our proofs.

Lemma 2.1. (Gronwall-Bellman) *Let an absolutely continuous function* $a : [\tau, T] \to \mathbb{R}$ *satisfy*

$$\dot{a}(t) \leq \lambda a(t) + b(t), \qquad \textit{for a.e. } t \in [\tau, T],$$

where $\tau \leq T$ *and* $\lambda \in \mathbb{R}$ *are constants, and* $b : [\tau, T] \to \mathbb{R}$ *is an integrable function. Then*

$$a(t) \leq e^{\lambda t} a(\tau) + \int_\tau^t e^{\lambda(t-s)} b(s) ds, \qquad \textit{for all } t \in [\tau, T].$$

Theorem 2.2. *Let $C : \mathbb{R} \rightrightarrows \mathbb{R}^n$ be a Lipschitz continuous function with constant L_C and let $C(t)$ be nonempty, closed and η-prox-regular for each $t \in \mathbb{R}$. Let $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy Lipschitz condition (5). Then the sweeping process (1) has at least one solution defined on the entire $\mathbb{R}$.*

The proof follows same steps as in the proofs of Theorem 2.1 and Theorem 2.2 of [14]. But we include the proof in the Appendix for completeness of the paper.

Theorem 2.3. *Let the conditions of Theorem 2.2 hold and $L_C \geq 0$ is as given by Theorem 2.2. Let*

$$\|f(t,x)\| \leq M_f, \ \text{for all } t \in \mathbb{R}, \ x \in \bigcup_{t\in\mathbb{R}} C(t), \tag{7}$$

where $M_f \geq 0$ is a fixed constant. Assume that f satisfies the strong monotonicity assumption (2) with

$$\alpha > \frac{L_C + M_f}{\eta}. \tag{8}$$

Then the sweeping process (1) has a unique solution $t \mapsto x(t)$, defined on $\mathbb{R}$. Furthermore the global solution $\mapsto x(t)$ is globally exponentially stable.

A similar to (8) condition has been earlier offered in Tanwani et al [26, Formula (3.5)] for the case $f(t, x) = -Ax$. It says that closer η to ∞ (closer the η-prox-regular set to a convex set) larger the interval of eligible α is. In particular, the case $\eta = \infty$ recovers the convex case, where it is sufficient to assume that $\alpha > 0$ (see [14, Theorem 2.2]).

Proof. We note that by Edmond-Thibault [12, Proposition 1] for a solution x of (1) with the initial condition $x(\tau) = x_0$,

$$\|\dot{x}(t) + f(t, x(t)\| \leq \|f(t, x(t)\| + L_C, \quad \text{for } t > \tau.$$

Then with uniform boundedness of f we have

$$\|\dot{x}(t) + f(t, x(t)\| \leq M_f + L_C, \quad \text{for } t > \tau. \tag{9}$$

Now let x_1, x_2 be two solutions of (1) with initial conditions $x_1(\tau), x_2(\tau) \in C(\tau)$. Let $t \geq \tau$ such that $x_1(t), x_2(t)$ are defined on $[t, \tau]$ and both $\dot{x}_1(t)$ and $\dot{x}_2(t)$ exist.
Since

$$-\dot{x}_1(t) - f(t, x_1(t)) \in N(C(t), (x_1(t)), \quad -\dot{x}_2(t) - f(t, x_2(t)) \in N(C(t), (x_2(t)),$$

by hypomonotonicity condition (6) of the normal cone and by (9) we have

$$\left\langle \frac{-\eta}{M_f + L_C}(\dot{x}_1(t) + f(t, x_1(t))) - \frac{-\eta}{M_f + L_C}(\dot{x}_2(t) + f(t, x_2(t))), x_1(t) - x_2(t) \right\rangle \geq -\|x_1(t) - x_2(t)\|^2.$$

Then

$$\|x_1(t) - x_2(t)\|^2 - \frac{\eta}{M_f + L_C}\langle f(t, x_1(t)) - f(t, x_2(t)), x_1(t) - x_2(t)\rangle \geq \frac{\eta}{M_f + L_C}\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle,$$

and by (2),

$$\frac{\eta}{M_f + L_C}\langle \dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)\rangle \leq \|x_1(t) - x_2(t)\|^2 - \frac{\eta\alpha}{M_f + L_C}\|x_1(t) - x_2(t)\|^2.$$

Thus we have

$$\langle \dot{x_1}(t) - \dot{x_2}(t), x_1(t) - x_2(t)\rangle \leq \left(\frac{M_f + L_C}{\eta} - \alpha\right) \|x_1(t) - x_2(t)\|^2,$$

i.e.

$$\frac{d}{dt}\|x_1(t) - x_2(t)\|^2 \leq \left(\frac{2(M_f + L_C)}{\eta} - 2\alpha\right) \|x_1(t) - x_2(t)\|^2.$$

Let $\bar{\alpha} = \frac{1}{\eta}(M_f + L_C - \eta\alpha)$. Then by Gronwall-Bellman lemma (2.1), for $t > \tau$,

$$\|x_1(t) - x_2(t)\|^2 \leq e^{2\bar{\alpha}(t-\tau)}\|x_1(\tau) - x_2(\tau)\|^2,$$

and so

$$\|x_1(t) - x_2(t)\| \leq e^{\bar{\alpha}(t-\tau)}\|x_1(\tau) - x_2(\tau)\|, \quad \text{for} \quad t > \tau. \tag{10}$$

Let $x(t)$ be a global solution of (1) which exists by Theorem 2.2. Then (8) guarantees that $\bar{\alpha} < 0$ and that $x(t)$ is exponentially stable. It remains to observe that $x(t)$ is the only global solution. Indeed, let $\bar{x}(t)$ be another global solution. Then, for each $t \in \mathbb{R}$ we can pass to the limit as $\tau \to -\infty$ in (10), obtaining $\|x(t) - \bar{x}(t)\| \leq 0$, so $x = \bar{x}$. □

Now we give a theorem about periodicity of the unique global solution established in Theorem 2.3. The proof follows the lines of Castaing and Monteiro Marques [8, Theorem 5.3], but we include such a proof for completeness.

Theorem 2.4. *The unique global solution x_0 which comes from Theorem 2.3 is T-periodic, if both maps $t \mapsto C(t)$ and $t \mapsto f(t, x)$ are T-periodic.*

Proof. Note that $a \mapsto x_a(T)$ is a contraction mapping from $C(0)$ to $C(T) = C(0)$, where x_a is the solution of (1) on $[0, T]$ with initial condition $x_a(0) = a \in C(0)$. Indeed, by (10), for $a, b \in C(0)$,

$$\|x_a(T) - x_b(T)\| \leq e^{\bar{\alpha}T}\|a - b\|$$

where $\bar{\alpha} < 0$.

Then, since $a \mapsto x_a(T)$ is continuous on $C(0)$ (see Edmond-Thibault [12, Proposition 2]), by the contraction mapping principle on $C(0)$ (see Rudin [25, p.220]), there exists $\bar{x} : [0, T] \to C(0)$ such that $\bar{x}(0) = \bar{x}(T)$ and satisfies (1) on $[0, T]$. Since both $t \mapsto C(t)$ and $t \mapsto f(t, x)$ are T-periodic, we can extend $\bar{x}$ to a T-periodic solution defined on $\mathbb{R}$ by T-periodicity.

Since the global solution x_0 given by Theorem 2.3 is unique, we have the result. □

3. A toy model.

In this section we consider an example where an r-prox-regular set is obtained as the complement of an ellipse to a circle, and where the strongly monotone vector field is just linear. We believe we introduce the simplest situation where the r-prox-regular set under consideration approaches a convex set when a parameter reaches certain critical value (being the height of the ellipse in our case). We call our example a toy model, because it seems to be of a benchmark value.

Let the vector field $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(t, x) := \alpha x, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^2, \tag{11}$$

where $\alpha > 0$ is a fixed constant. We define the moving set $C(t)$ using a function $b \in C^1(\mathbb{R}, \mathbb{R})$ which is bounded below by $\beta \geq 1$ and admits a global Lipschitz constant L_b, i.e.

$$|b(t_1) - b(t_2)| \leq L_b|t_1 - t_2|, \quad \text{for all} \quad t_1, t_2 \in \mathbb{R}. \tag{12}$$

Define

$$C(t) := \bar{B}_1 \bigcap S(t), \quad S(t) = \left\{ x \in \mathbb{R}^2 : x_1^2 + \frac{x_2^2}{b(t)^2} \geq 1 \right\}. \tag{13}$$

where $\bar{B}_1$ is the closed ball of radius 1 and centered at $(-1.5, 0)$.

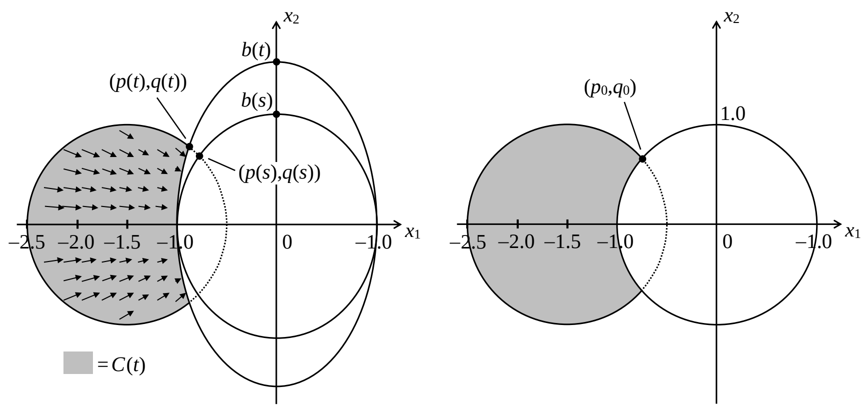

FIGURE 1. Illustrations of the notations of the example. The closed ball centered at $(-1.5, 0)$ is $\bar{B}_1$ and the white ellipses are the graphs of $S(t)$ for different values of the argument. The arrows is the vector field of $\dot{x} = -\alpha x$.

In order to apply Theorem 2.3, we will now analyze: *i)* strong monotonicity and uniform boundedness of $f(t, x)$, *ii)* Lipschitz continuity of $C(t)$, *iii)* prox-regularity of $C(t)$.

i) The monotonicity and boundedness of $f(t, x)$. Since $\langle f(t, x) - f(t, y), x - y \rangle = \langle \alpha x - \alpha y, x - y \rangle = \alpha \|x - y\|^2$, f is strongly monotone with constant α and bounded on $\bar{B}_1 \supset C(t)$ by $M_f = 2.5\alpha$.

ii) Lipschitz continuity of $C(t)$. The boundary $\partial \bar{B}_1$ of $\bar{B}_1$ intersects the boundary $\partial S(t)$ of $S(t)$ at a unique point $(p(t), q(t))$ with $q(t) \geq 0$. Since

$$d_H(C(t), C(s)) \leq \|(p(t), q(t)) - (p(s), q(s))\|$$

(see Fig. 1), we now aim at computing the Lipschitz constants of functions p and q. Since $b \in C^1(\mathbb{R}, [1, \infty))$, the implicit function theorem (see e.g. Zorich [29, Sec. 8.5.4, Theorem 1]) ensures that p and q are differentiable on $\mathbb{R}$. Therefore, by the mean-value theorem (see e.g. Rudin [25, Theorem 5.10]),

$$d_H(C(t), C(s)) \leq \|(p'(t_p), q'(t_q))\| \cdot |t - s|, \tag{14}$$

where t_p, t_q are located between t and s. To compute $(p'(t_p), q'(t_q))$, we use the formula for the derivative of the implicit function (Zorich [29, Sec. 8.5.4, Theorem 1])

$$(p'(t), q'(t))^T = -\left(F'_{(p,q)}\right)^{-1} (p(t), q(t), t) F'_t(p(t), q(t), t),$$

applied with

$$F(p, q, t) = \begin{pmatrix} (p + 1.5)^2 + q^2 - 1 \\ p^2 + \dfrac{q^2}{b(t)^2} - 1 \end{pmatrix}.$$

Since

$$F'_{(p,q)}(p,q,t) = 2\begin{pmatrix} p+1.5 & q \\ p & \dfrac{q}{b(t)^2} \end{pmatrix}, \quad F'_t(p,q,t) = \begin{pmatrix} 0 \\ -2b(t)^{-3}b'(t)q^2 \end{pmatrix},$$

we get the following formula for the derivatives p' and q'

$$\begin{pmatrix} p'(t) \\ q'(t) \end{pmatrix} = -\frac{1}{\dfrac{1}{b(t)^2}(p(t)+1.5)q(t) - p(t)q(t)} \begin{pmatrix} q(t) \\ -(p(t)+1.5) \end{pmatrix} \frac{1}{b(t)^3} q(t)^2 b'(t).$$

Noticing that the properties $1+p(t)>0$ and $-p(t)b(t)^2>0$ imply

$$\frac{1}{b(t)\cdot(p(t)+1.5-p(t)b(t)^2)} \le \frac{1}{\beta\cdot(-p(t)b(t)^2)} \le \frac{1}{\beta^3|p_0|},$$

we conclude

$$|p'(t)| \le \frac{L_b}{\beta^3|p_0|}, \qquad |q'(t)| \le \frac{L_b}{\beta^3|p_0|},$$

where p_0 is such that $p(t)\le p_0$ for all $t\in\mathbb{R}$. Since $b(t)\ge 1$, we can take p_0 as the abscissa of the intersection of $\partial\bar{B}_1$ with a unit circle centered at 0, i.e.

$$p_0 = -0.75,$$

see Fig. 1. Substituting these achievements to (14), we conclude

$$d_H(C(t),C(s)) \le \frac{4L_b}{3\beta^3}|t-s|,$$

which gives $L_C = \dfrac{4L_b}{3\beta^3}$ for the Lipschitz constant of $t\mapsto C(t)$.

iii) The constant η in η-prox-regularity of $C(t)$. We recall that $C(t)$ is η-prox-regular if $C(t)$ admits an external tangent ball with radius smaller than η at each $x\in\partial C(t)$ (see Poliquin et al [22], Maury and Venel [20], Colombo and Monteiro Marques [10]). The points of $\partial C(t)\backslash\partial S(t)$ admit an external tangent ball of any radius. Therefore, to find η, which determines η-prox-regularity of $C(t)$, it is sufficient to focus on the points of $\partial C(t)\cap\partial S(t)$. That is why, for a fixed $t\in\mathbb{R}$, we can choose η as the minimum of the radius of curvature through $x\in\partial C(t)\cap\partial S(t)$, see e.g. Lockwood [19, p. 193].

Let us fix $t\in\mathbb{R}$ and use the parameterization $P(\phi)=(-\cos\phi, b(t)\sin\phi)$, $\phi\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, for the left-hand side of the ellipse $x^2+\frac{y^2}{b(t)^2}=1$. Then, the radius of curvature $R(\phi)$ of $\partial C(t)\cap\partial S(t)$ at $P(\phi)$ is (see Lockwood [19, p. xi, p. 21])

$$R(\phi) = \frac{1}{b(t)}(\sin^2\phi + b(t)^2\cos^2\phi)^{\frac{3}{2}} = \frac{1}{b(t)}\left(b(t)^2 + (1-b(t)^2)\sin^2(\phi)\right)^{\frac{3}{2}}.$$

Observe that R decreases when $|\phi|$ increases from 0 to $\dfrac{\pi}{2}$.

Therefore, the minimum curvature of $\partial C(t)\cap\partial S(t)$ is attained at the point $(p(t),q(t))$ as defined in ii). Let ϕ_0 be such that $P(\phi_0)=(p(t),q(t))$ and let $\phi_*>0$ be such that the second component $P_2(\phi_*)$ of $P(\phi_*)$ equals 1, which exists because $b(t)\ge 1$ (see Fig. 2). Since $q(t)\le 1$, we have $\phi_0\le\phi_*$, and since $\phi\to R(\phi)$ decreases as $|\phi|$ increases, we have

$$R(\phi_0)\ge R(\phi_*).$$

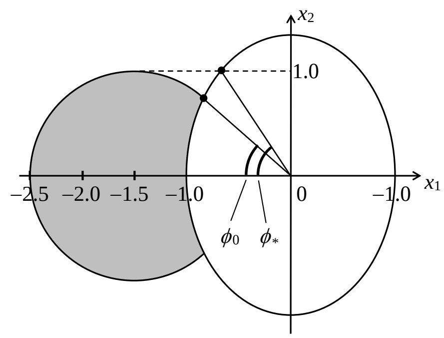

FIGURE 2. The parameters ϕ_0 and ϕ_*.

Since $P_2(\phi_*) = 1$ implies $b(t)\sin\phi_* = 1$, we have $\sin\phi_* = \dfrac{1}{b(t)}$ and so

$$\begin{aligned} R(\phi_0) \quad &\geq \quad \frac{1}{b(t)}\left(\frac{1}{b(t)^2} + b(t)^2\left(1 - \frac{1}{b(t)^2}\right)\right)^{\frac{3}{2}} = \frac{1}{b(t)} \cdot \frac{(1 + b(t)^4 - b(t)^2)^{\frac{3}{2}}}{b(t)^3} = \\ &= \left(b(t)^{-\frac{8}{3}} + b(t)^{\frac{4}{3}} - b(t)^{-\frac{2}{3}}\right)^{\frac{3}{2}} \geq \left(b(t)^{\frac{4}{3}} - b(t)^{-\frac{2}{3}}\right)^{\frac{3}{2}}. \end{aligned}$$

Noticing that the function $b \mapsto \left(b^{\frac{4}{3}} - b^{-\frac{2}{3}}\right)^{\frac{3}{2}}$ increases on $[1, \infty)$, we finally conclude

$$R(\phi_0) \geq \left(\beta^{\frac{4}{3}} - \beta^{-\frac{2}{3}}\right)^{\frac{3}{2}}.$$

Therefore, $C(t)$ is η-prox-regular with $\eta = \left(\beta^{\frac{4}{3}} - \beta^{-\frac{2}{3}}\right)^{\frac{3}{2}}$.

Substituting the values of M_f, L_C, and η into formula (8), we get the following statement.

Proposition 1. *Let $\alpha > 0$ be an arbitrary constant and $b \in C^1(\mathbb{R}, [\beta, \infty))$ with some $\beta \geq 1$ and Lipschitz condition (12). If*

$$\alpha > \frac{\dfrac{4L_b}{3\beta^3} + \dfrac{5}{2}\alpha}{\left(\beta^{\frac{4}{3}} - \beta^{-\frac{2}{3}}\right)^{\frac{3}{2}}},$$

then, the global solution

$$x(t) = (-1, 0), \quad t \in \mathbb{R},$$

of the sweeping process (1) with $C(t)$ and $f(t,x)$ given by (13) and (11), is globally asymptotically stable.

As noticed earlier, $b \mapsto \left(b^{\frac{4}{3}} - b^{-\frac{2}{3}}\right)^{\frac{3}{2}}$ increases on $[1, \infty)$, so that the condition of Proposition 1 is a lower bound on β.

4. **Conclusion.** In this paper we proved the existence of at least one global solution to a nonconvex sweeping process with Lipschitz perturbations. The uniqueness and exponential stability of the solution follows when the vector field of the sweeping process is uniformly bounded, strongly monotone and the prox-regularity constant

of the moving constraint is not too small. A similar condition has been earlier introduced in Tanwani et al [26, Formula (3.5)] where the case of a linear perturbation is addressed. We further proved that the unique global solution is periodic when the right-hand-sides of the sweeping process are periodic in time.

Following the lines of Kamenskii et al [14], the ideas of the present work can be extended to almost periodic solutions and to sweeping processes with small non-monotone ingredients.

We show in Appendix that the estimate for the prox-regularity constant in Maury-Venel [20, Proposition 2.15, Proposition 2.17] does not agree with inequality (8), making our main result inapplicable to the model of [20]. At the same time, we analyze a toy model where we document how applicability or inapplicability of our result is linked to the parameters of sweeping process.

The ultimate conclusion of the paper agrees with that of Tanwani et al [26]: closer the constraint to a convex one, weaker monotonicity is required to keep the sweeping process globally asymptotically stable.

5. Appendix.

5.1. Proof of Theorem 2.2. Let $\{\xi_n\}_{n=1}^{\infty} \subset \mathbb{R}^n$ be such that $\xi_n \in C(-n)$ for each $n \in \mathbb{N}$. Define

$$x_n(t) = \begin{cases} x(t, -n, \xi_n) & \text{if } t \geq -n \\ \xi_n & \text{if } t < -n \end{cases}$$

where $t \mapsto x(t, -n, \xi_n)$ is the solution $t \mapsto x(t)$ of (1) with the initial condition $x(-n) = \xi_n$, $n \in \mathbb{N}$. Since $C(t)$ is globally bounded, then, given any $k \in \mathbb{N}$, Edmond-Thibault [12, Theorem 1] ensures that all solutions of the sweeping process (1) on the interval $[-k, k]$ share the same Lipschitz constant $L_k > 0$.

We now follow the standard diagonal process in order to extract such a subsequence from $\{x_n(t)\}_{n=1}^{\infty}$ which convergences uniformly on any interval $[-k, k]$, $k \in \mathbb{N}$. Let us denote $x_n^0(t) = x_n(t)$ on $\mathbb{R}$ for each $n \in \mathbb{N}$. By Arzela-Ascoli theorem, there exists a subsequence $\{x_n^1(t)\}_{n=1}^{\infty} \subset \{x_n^0(t)\}_{n=1}^{\infty}$ which converges uniformly on $[-1, 1]$. Analogously, there exists a subsequence $\{x_n^2(t)\}_{n=1}^{\infty} \subset \{x_n^1(t)\}_{n=1}^{\infty}$ which converges uniformly on $[-2, 2]$. Repeating this procedure infinitely, we get a family of subsequences $\{x_n^k(t)\}_{n=1}^{\infty} \subset \{x_n^{k-1}(t)\}_{n=1}^{\infty}$, $k \in \mathbb{N}$, such that $\{x_n^k(t)\}_{n=1}^{\infty}$ converges uniformly on $[-k, k]$ for any $k \in \mathbb{N}$. Defining $\bar{x}_n(t) = x_n^n(t)$ on $\mathbb{R}$ for each $n \in \mathbb{N}$, we get that $\{\bar{x}_n(t)\}_{n=1}^{\infty}$ converges uniformly on any $[-k, k]$, $k \in \mathbb{N}$. Let $\bar{x}(t) = \lim_{n\to\infty} \bar{x}_n(t)$.

Let us now show that $\bar{x}(t)$ is a solution of the sweeping process (1). Denote by $x(t)$ a solution of (1) with the initial condition $x(\tau) = \bar{x}(\tau)$. Assume $x(t_0) \neq \bar{x}(t_0)$ for some $t_0 > \tau$. i.e. $x(t_0) \neq \lim_{n\to\infty} \bar{x}_n(t_0)$. Then there exist $\varepsilon_0 > 0$, such that

$$\text{for each } n \in \mathbb{N}, \text{ there exists } m_n > n \text{ such that } \|x(t_0) - \bar{x}_{m_n}(t_0)\| \geq \varepsilon_0. \quad (15)$$

Recall, $\bar{x}_{m_n}(t)$ is a solution of (1) for $t \geq -m_n$. Therefore, we can use the continuous dependence of solutions on the initial condition (see Edmond-Thibault [12, Proposition 2]) to conclude the existence of $\delta > 0$ such that

$$\text{if } \tau \geq -m_n \text{ and } \|x(\tau) - \bar{x}_{m_n}(\tau)\| < \delta \text{ then } \|x(t) - \bar{x}_{m_n}(t)\| < \varepsilon_0,\ t \in [\tau, t_0]. \quad (16)$$

The statements (15) and (16) contradict each other for $n \in \mathbb{N}$ sufficiently large. Therefore, $x(t) = \bar{x}(t)$ for all $t \geq \tau$, i.e. $\bar{x}(t)$ is a solution of (1). □

5.2. **The crowd motion model.** We give a brief introduction into the model by Maury-Venel [20], before we explain the inapplicability of Theorem 2.3 in this model.

Consider N people with positions given by $x = (x_1, x_2, \dots, x_N)$, where each person is geometrically represented as a disk with center $x_i \in \mathbb{R}^2$ and radius r, so that $x \in \mathbb{R}^{2N}$. Two people (say i-th and j-th) cannot overlap, therefore we have an unilateral constraint $\|x_i - x_j\| \geqslant 2r$ and so the set of feasible configurations is defined as (see [20])

$$C = \{x \in \mathbb{R}^{2N} : \|x_i - x_j\| - 2r \geq 0 \text{ for all } i < j\}. \tag{17}$$

Let $U(x) = (U_1(x), U_2(x), \cdots U_N(x))$ be the spontaneous velocity of each person at the position x, i.e. $U_i(x)$ is the velocity that i-th person would have in the absence of other people. Since the aim of Maury-Venel [20] is to have a model that describes people in a highly packed situation, the actual velocity of a person is defined to be closest to the spontaneous velocity. So the actual velocity is computed as the projection of the spontaneous velocity onto the set of feasible velocities. This gives the sweeping process (see [20])

$$\begin{cases} -\dot{x} \in N(C, x) - U(x) \\ x(0) = x_0 \in C. \end{cases} \tag{18}$$

Let us consider the situation where there are only two people. By Maury-Venel [20, Proposition 2.15], the set C in (17) is η-prox regular with $\eta = r\sqrt{2}$. Let us take $U(x) = -x$. Viewing (18) as (1), we get $f(t, x) = x$ and so $\alpha = 1$ in (2). Then condition (8) of Theorem 2.3 takes the form $\sqrt{2}r > L_C + M_f$, where $L_C = 0$ (because C in (18) does not depend on t). Therefore, (8) implies $M_f < \sqrt{2}r$. On the other hand, according to (7), M_f must satisfy $M_f \geqslant \|f(t, x)\|$ with $f(t, x) = x$ for each $x \in C$. Let us consider a pair of people positioned at $(0, -r)$ and $(0, r)$. Since $\|(0, -r) - (0, r)\| = 2r$, we have $(0, -r, 0, r) \in C$ and so M_f verifies $M_f \geqslant \|f(0, -r, 0, r)\| = \|(0, -r, 0, r)\| = \sqrt{0^2 + (-r)^2 + 0^2 + r^2} = \sqrt{2}r$. Therefore, Theorem 2.3 does not apply.

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REFERENCES

[1] L. Adam and J. Outrata, On optimal control of a sweeping process coupled with an ordinary differential equation, *Discrete Contin. Dyn. Syst.–Ser. B*, **19** (2014), 2709–2738.

[2] J. Bastien, F. Bernardin and C.-H. Lamarque, *Non Smooth Deterministic or Stochastic Discrete,* Dynamical Systems: Applications to Models with Friction or Impact, Wiley, 2013, 512 pp.

[3] H. Benabdellah, Existence of solutions to the nonconvex sweeping process, *Journal of Differential Equations*, **164** (2000), 286–295.

[4] B. Brogliato, Absolute stability and the Lagrange–Dirichlet theorem with monotone multivalued mappings, *Systems & Control Letters*, **51** (2004), 343–353.

[5] B. Brogliato and W. M. H. Heemels, Observer design for Lur'e systems with multivalued mappings: A passivity approach, *IEEE Transactions on Automatic Control*, **54** (2009), 1996–2001.

[6] T. H. Cao and B. S. Mordukhovich, Optimality conditions for a controlled sweeping process with applications to the crowd motion model, *Discrete Cont. Dyn. Syst., Ser B.*, **22** (2017), 267–306.

[7] T. H. Cao and B. Mordukhovich, Optimal control of a nonconvex perturbed sweeping process, *Journal of Differential Equations*, **266** (2019), 1003–1050.

[8] C. Castaing and M. D. Monteiro Marques, BV periodic solutions of an evolution problem associated with continuous moving convex sets, *Set-Valued Analysis*, **3** (1995), 381–399.

[9] G. Colombo and V. V. Goncharov, The sweeping processes without convexity, *Set-Valued Analysis*, **7** (1999), 357–374.

[10] G. Colombo and M. D. Monteiro Marques, Sweeping by a continuous prox-regular set, *Journal of Differential Equations*, **187** (2003), 46–62.

[11] J. F. Edmond and L. Thibault, BV solutions of nonconvex sweeping process differential inclusion with perturbation, *Journal of Differential Equations*, **226** (2006), 135–179.

[12] J. F. Edmond and L. Thibault, Relaxation of an optimal control problem involving a perturbed sweeping process, *Mathematical Programming*, **104** (2005), 347–373.

[13] C. O. Frederick and P. J. Armstrong, Convergent internal stresses and steady cyclic states of stress, *The Journal of Strain Analysis for Engineering Design*, **1** (1966), 154–159.

[14] M. Kamenskii, O. Makarenkov, L. N. Wadippuli, and P. R. de Fitte, Global stability of almost periodic solutions of monotone sweeping processes and their response to non-monotone perturbations, *Nonlinear Analysis: Hybrid Systems*, **30** (2018), 213–224.

[15] M. Kamenskii and O. Makarenkov, On the response of autonomous sweeping processes to periodic perturbations, *Set-Valued and Variational Analysis*, **24** (2016), 551–563.

[16] P. Krejci, *Hysteresis, Convexity and Dissipation in Hyperbolic Equations*, Gattotoscho, 1996.

[17] M. Kunze, Periodic solutions of non-linear kinematic hardening models, *Math. Methods Appl. Sci.*, **22** (1999), 515–529.

[18] R. I. Leine and N. Van de Wouw, *Stability and Convergence of Mechanical Systems with Unilateral Constraints*, Lecture Notes in Applied and Computational Mechanics, 36. Springer-Verlag, Berlin, 2008.

[19] E. H. Lockwood, *A Book of Curves*, Cambridge University Press, New York, 1961.

[20] B. Maury and J. Venel, A discrete contact model for crowd motion, *ESAIM: Mathematical Modelling and Numerical Analysis*, **45** (2011), 145–168.

[21] B. S. Mordukhovich, *Variational Analysis and Applications*, Springer, 2018.

[22] R. A. Poliquin, R. T. Rockafellar, L. Thibault, Local differentiability of distance functions, *Transactions of American Mathematical Society*, **352** (2000), 5231–5249.

[23] C. Polizzotto, Variational methods for the steady state response of elasticplastic solids subjected to cyclic loads, *International Journal of Solids and Structures*, **40** (2003), 2673–2697.

[24] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, Berlin, 1998.

[25] W. Rudin, *Principles of Mathematical Analysis*, McGraw-hill New York, 1976.

[26] A. Tanwani, B. Brogliato and C. Prieur, Stability and observer design for Lur'e systems with multivalued, nonmonotone, time-varying nonlinearities and state jumps, *SIAM Journal on Control and Optimization*, **52** (2014), 3639–3672.

[27] L. Thibault, Sweeping process with regular and nonregular sets, *Journal of Differential Equations*, **193** (2003), 1–26.

[28] Y. V. Trubnikov and A. I. Perov, *Differential Equations with Monotone Nonlinearities,* "Nauka i Tekhnika", Minsk, 1986.

[29] V. A. Zorich, *Mathematical Analysis. II*, Translated from the 2002 fourth Russian edition by Roger Cooke, Universitext, Springer-Verlag, Berlin, 2004.

[30] Z. Zhu, H. Leung and Z. Ding, Optimal synchronization of chaotic systems in noise, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, **46** (1999), 1320–1329.

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